MA 241, Spring 2023, TEST 4 answers

Problems 1 and 2

These were in different orders in different versions of the tests.

(1 or 2). Consider the sequence $\{a_n\}$ defined by $a_n = \dfrac{8n}{2^n} - n$.

a. Give the first several terms of the sequence:

$a_1 = 3 \qquad a_2 = 2 \qquad a_3 = 0 \qquad a_4 = -2$

b. The sequence is increasing $\boxed{\text{decreasing}}$ not monotonic.

(2 or 1). Consider the sequence $\{a_n\}$ defined by $a_1 = 12$, $a_2 = 4$, and $a_n = \dfrac{a_{n-2} - a_{n-1}}{2}$.

Give the first several terms of the sequence

$a_1 = 12 \qquad a_2 = 4 \qquad a_3 = 4 \qquad a_4 = 0$

Other versions of the problem had different values for a_1 and a_2. So your answers might have been:

$a_1 = 10 \qquad a_2 = 6 \qquad a_3 = 2 \qquad a_4 = 2$

$a_1 = 8 \qquad a_2 = 4 \qquad a_3 = 2 \qquad a_4 = 1$

$a_1 = 16 \qquad a_2 = 8 \qquad a_3 = 4 \qquad a_4 = 2$

Problem 3

3. Determine if the sequence $\{a_n\}$ defined by $a_n = (-1)^n \dfrac{3n^2}{n^2+1}$ converges or diverges. If it converges, give the limit.

It diverges. The sequence $\dfrac{3n^2}{n^2+1}$ converges to 3, so in the limit, the sequence a_n bounces back and forth between close-to-3 and close-to-(-3). You may have had a number other than 3 in the numerator, but the explanation is the same.

Problem 4

4. Use series techniques to write the repeating decimal $0.\overline{657}$ as a fraction. You don't have to reduce your fraction, but at least you must simplify it to $\frac{\text{integer}}{\text{integer}}$.

$$0.\overline{657} = \frac{657}{1000} + \frac{657}{1000000} + \cdots$$

This is a geometric series with first term $a = \frac{657}{1000}$ and ratio $r = \frac{1}{1000}$. Since $|r| < 1$, it converges to

$$\frac{a}{1-r} = \frac{\frac{657}{1000}}{1 - \frac{1}{1000}} = \frac{\frac{657}{1000}}{\frac{999}{1000}} = \frac{657}{999} = \frac{73}{111}$$

Your problem may have been different, but the method was the same. You may have had:

$$0.\overline{873} = \tfrac{873}{999} = \tfrac{93}{111}$$

$$0.\overline{423} = \tfrac{423}{999} = \tfrac{47}{111}$$

$$0.\overline{261} = \tfrac{261}{999} = \tfrac{29}{111}$$

Problem 5

The order of the problems varied and there were tiny variations in the problems, but the methods and results were the same.

5. For each of these series, determine if the series converges or diverges. If it convergent **and if you can give an exact sum**, determine the sum.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2+1}$ Converges.

Method 1: Comparison Test.
Since $0 < \frac{6}{n^2+1} < \frac{6}{n^2}$ for all $n \geq 1$ and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it is a constant times a p-series for $p = 2$), the Comparison Test says that $\sum_{n=1}^{\infty} \frac{6}{n^2+1}$ converges.

Method 2: Limit Comparison Test.
Calculate $\displaystyle\lim_{n\to\infty} \frac{\frac{6}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{6n^2}{n^2+1} = 6$. Since this limit is greater than zero (and not "infinity"), the series $\sum_{n=1}^{\infty} \frac{6}{n^2+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ either both converge or both diverge. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2$). So $\sum_{n=1}^{\infty} \frac{6}{n^2+1}$ converges as well.

Method 3: Integral Test.
We compute $\int_1^{\infty} \frac{6}{x^2+1}\,\mathrm{d}x = \lim_{b\to\infty}[6\arctan x]_1^b = 3\pi - \frac{3}{2}\pi = \frac{3}{2}\pi$. Since this integral converges, $\sum_{n=1}^{\infty} \frac{6}{n^2+1}$ converges by the Integral Test.

b. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-1}$ Converges.

Method 1: Limit Comparison Test.
Calculate $\displaystyle\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2-1}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$. Since this limit is greater than zero (and not "infinity"), the series

$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-1}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ either both converge or both diverge. We know that $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges (p-series with $p = \frac{3}{2}$). So $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-1}$ converges as well.

Other Methods:
Integral Test: You can get the Integral Test to work, but the easiest way I see to do that integral involves a substitution $u = \sqrt{x}$ and then a partial fractions computation. Then you have to think a bit about convergence of the integral.
Comparison Test: You can probably get this to work too, but if you compare with $\frac{1}{n^{\frac{3}{2}}}$, it is wrong, because the comparison goes in the wrong direction.

c. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n^{1.2}} - \left(\frac{5}{4}\right)^n \right)$ Diverges.

$\sum_{n=1}^{\infty} \frac{2}{n^{1.2}}$ converges (p-series test, $p = 1.1$) and $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n$ diverges (Test for Divergence, or geometric series with $|r| \geq 1$). Thus $\sum_{n=1}^{\infty} \left(\frac{2}{n^{1.2}} - \left(\frac{5}{4}\right)^n\right)$ also diverges.

d. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 3^n}{8^n}$ Converges (and there is an exact answer).

There were several versions of the problem. In each case, you broke into two series and noticed that each is a convergent geometric series. Your answer was one of the following:

$$\sum_{n=1}^{\infty} \frac{2^n - 3^n}{8^n} = \sum_{n=1}^{\infty} \frac{2^n}{8^n} - \sum_{n=1}^{\infty} \frac{3^n}{8^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n - \sum_{n=1}^{\infty} \left(\frac{3}{8}\right)^n = \frac{\frac{1}{4}}{1-\frac{1}{4}} - \frac{\frac{3}{8}}{1-\frac{3}{8}} = \frac{1}{3} - \frac{3}{5} = -\frac{4}{15}$$
$$\sum_{n=1}^{\infty} \frac{2^n - 3^n}{9^n} = \sum_{n=1}^{\infty} \frac{2^n}{9^n} - \sum_{n=1}^{\infty} \frac{3^n}{9^n} = \sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{\frac{2}{9}}{1-\frac{2}{9}} - \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{2}{7} - \frac{1}{2} = -\frac{3}{14}$$
$$\sum_{n=1}^{\infty} \frac{2^n - 3^n}{6^n} = \sum_{n=1}^{\infty} \frac{2^n}{6^n} - \sum_{n=1}^{\infty} \frac{3^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{3}}{1-\frac{1}{3}} - \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2} - 1 = -\frac{1}{2}$$
$$\sum_{n=1}^{\infty} \frac{2^n - 3^n}{5^n} = \sum_{n=1}^{\infty} \frac{2^n}{5^n} - \sum_{n=1}^{\infty} \frac{3^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n - \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \frac{\frac{2}{5}}{1-\frac{2}{5}} - \frac{\frac{3}{5}}{1-\frac{3}{5}} = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$$

Problem 6

Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n \cdot b^n}$ converges or diverges. If it convergent **and if you can give an exact value**, determine the sum. (The number b in your problem was 3, 4, 6, or 7.)

For any version of the problem, you can do the Ratio Test. You get

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{5^{n+1}}{(n+1)\cdot b^{n+1}}}{\frac{5^n}{n \cdot b^n}} = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)\cdot b^{n+1}} \cdot \frac{n \cdot b^n}{5^n} = \lim_{n\to\infty} \frac{5n}{(n+1)\cdot b} = \frac{5}{b}$$

For $b > 5$ (if your b was 6 or 7), the series converges by the Ratio Test. Or you can prove convergence by the Comparison Test (comparing to $\sum_{n=1}^{\infty} (\frac{5}{b})^n$).
For $b < 5$ (if your b was 3 or 4), this diverges by the Ratio Test. Or you can prove divergence by the Comparison Test (comparing to $\sum_{n=1}^{\infty} \frac{1}{n}$).

Problem 7

7. Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[q]{n}}$ is absolutely convergent, conditionally convergent, or divergent. (Your q was 3, 4, 5, or 6.)

In any case, this is conditionally convergent. It converges by the Alternating Series Test, because it alternates and the absolute values $\dfrac{1}{\sqrt[q]{n}}$ of the terms decrease and limit to zero.

But the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[q]{n}}$ of the absolute values of the terms diverges by the p-test with $p = \frac{1}{q}$. So the series is not absolutely convergent.

Problem 8

8. Find the radius of convergence of each of the following series.

a. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \cdot c^n}$

Your c was 2, 3, 4, or 5, and you might not have had a $(-1)^n$ in your series, but that won't matter, because in the Ratio Test, we'll take absolute values.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{x^{n+1}}{(n+1)\cdot c^{n+1}}}{(-1)^n \frac{x^n}{n \cdot c^n}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cdot c^{n+1}} \cdot \frac{n \cdot c^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x \cdot n}{(n+1)\cdot c} \right| = \left| \frac{x}{c} \right|.$$

So the Ratio Test says that this converges (absolutely, but we don't need that) if $\left|\frac{x}{c}\right| < 1$ and diverges if $\left|\frac{x}{c}\right| > 1$. In other words, converges if $|x| < c$ and diverges if $|x| > 1$. That means the radius of convergence is c.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(kx)^n}{n!}$

Your k was 2, 3, 4, or 5, but k won't matter to the answer. Also, you might have had a $(-1)^n$ in your series, but that won't matter, because in the Ratio Test, we'll take absolute values.

$$\lim_{n\to\infty} \left| \frac{\frac{(kx)^{n+1}}{(n+1)!}}{\frac{(kx)^n}{n!}} \right| \lim_{n\to\infty} \left| \frac{(kx)^{n+1}}{(n+1)!} \cdot \frac{n!}{(kx)^n} \right| = \lim_{n\to\infty} \left| \frac{kx}{n+1} \right| = 0.$$

So the Ratio Test says that no matter what x is, this converges (absolutely, but we don't need that). That means the radius of convergence is ∞.

Challenge Problems

9. Write the **base 7** repeating "septimal" $0.\overline{42}$ as a reduced base-7 fraction.
   Just for fun, here is the entire thing in base 7 notation:

$$0.\overline{42} = \frac{42}{100} + \frac{42}{10000} + \cdots = \frac{\frac{42}{100}}{1 - \frac{1}{100}} = \frac{\frac{42}{100}}{\frac{66}{100}} = \frac{42}{66} = \frac{5}{11}.$$

If you didn't think that was fun, here it is in base 10 notation. The base-7 number 42 is $4*7+2=30$. The base-7 repeating "septimal" $0.\overline{42}$ is

$$\frac{30}{7^2}+\frac{30}{7^4}+\cdots=\frac{\frac{30}{49}}{1-\frac{1}{49}}=\frac{\frac{30}{49}}{\frac{48}{49}}=\frac{30}{48}=\frac{5}{8}.$$

In base-7, this is $\frac{5}{11}$.

10. One of our convergence tests says that the following series converges. Find the sum exactly.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{n(n+1)}=\frac{3}{2}-\frac{5}{6}+\frac{7}{12}-\frac{9}{20}+\frac{11}{30}-\cdots$$

Partial fractions gives $\frac{2n+1}{n(n+1)}=\frac{1}{n}+\frac{1}{n+1}$, so this series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{n(n+1)}=\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)-\left(\frac{1}{4}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{6}\right)-\left(\frac{1}{6}+\frac{1}{7}\right)+-\cdots$$

This is telescoping (in a slightly different way than we have seen before), and the partial sums limit to 1.

11. Suppose $f(x)$ and $g(x)$ are polynomials with positive coefficients. Formulate a rule that tells exactly when $\sum_{n=1}^{\infty}\frac{f(n)}{g(n)}$ converges. Justify your answer.

If you've done enough problems with the Limit Comparison Test (and if you don't mind thinking more abstractly about "polynomials" instead of specific formulas), you'll see that you should do the Limit Comparison Test with $\sum_{n=1}^{\infty}\frac{1}{n^p}$ for some p. Specifically, p should be the degree of $g(x)$ minus the degree of $f(x)$. The limit $\lim_{n\to\infty}\frac{\frac{f(n)}{g(n)}}{\frac{1}{n^p}}=\lim_{n\to\infty}\frac{f(n)\cdot n^p}{g(n)}$ is some positive number, so the two series either both converge or both diverge. Conclusion: $\sum_{n=1}^{\infty}\frac{f(n)}{g(n)}$ converges if $p<1$ and diverges if $p\geq 1$. Said another way: $\sum_{n=1}^{\infty}\frac{f(n)}{g(n)}$ converges if and only if the degree of the denominator is at least 2 greater than the degreee of the numerator.