MA 241 Spring 2023, Test 2 Answers

1. $\displaystyle\int \tan^4 x \sec^4 x \, \mathrm{d}x = \int \tan^4 x (\tan^2 x + 1) \sec^2 x \, \mathrm{d}x$

(Substitution: $u = \tan x \qquad \mathrm{d}u = \sec^2 x \, \mathrm{d}x$)

$\displaystyle = \int u^4(u^2+1)\,\mathrm{d}u = \int (u^6+u^4)\,\mathrm{d}u = \frac{1}{7}u^7 + \frac{1}{5}u^5 + C = \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

1. $\displaystyle\int \tan^2 x \sec^4 x \, \mathrm{d}x = \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, \mathrm{d}x$

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$\displaystyle = \int u^2(u^2+1)\,\mathrm{d}u = \int (u^4+u^2)\,\mathrm{d}u = \frac{1}{5}u^5 + \frac{1}{3}u^3 + C = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

1. $\displaystyle\int \tan^3 x \sec^7 x \, \mathrm{d}x = (\sec^2 x - 1) \sec^6 x \tan x \sec x \, \mathrm{d}x$

(Substitution: $u = \sec x \qquad \mathrm{d}u = \tan x \sec x \, \mathrm{d}x$)

$\displaystyle = \int (u^2-1)u^6\,\mathrm{d}u = \int (u^8-u^6)\,\mathrm{d}u = \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C$

1. $\displaystyle\int \tan^3 x \sec^5 x \, \mathrm{d}x = (\sec^2 x - 1) \sec^4 x \tan x \sec x \, \mathrm{d}x$

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$\displaystyle = \int (u^2-1)u^4\,\mathrm{d}u = \int (u^6-u^4)\,\mathrm{d}u = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C$

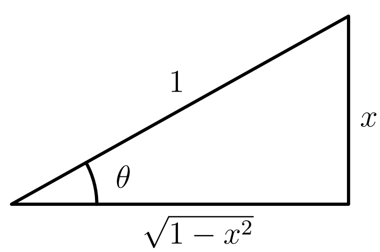
2. In your problem, a was 3, 5, 7, or 9.

$\displaystyle\int x^5 \left(\sqrt{1-x^2}\right)^a \mathrm{d}x$

$\displaystyle = \int \sin^5\theta \cos^a\theta (\cos\theta\,\mathrm{d}\theta)$

$\displaystyle = \int \sin^5\theta \cos^{a+1}\theta\,\mathrm{d}\theta$

$\displaystyle = -\int (1-\cos^2\theta)^2 \cos^{a+1}\theta (-\sin\theta\,\mathrm{d}\theta)$



$x = \sin\theta$

$\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$

$\sqrt{x^2-1} = \cos\theta$

(Substitution: $u = \cos x \qquad \mathrm{d}u = -\sin x \, \mathrm{d}x$)

$\displaystyle = -\int (1-u^2)^2 u^{a+1}\,\mathrm{d}u = \int (-u^{a+1} + 2u^{a+3} - u^{a+5})\,\mathrm{d}u = -\frac{1}{a+2}u^{a+2} + \frac{2}{a+4}u^{a+4} - \frac{1}{a+6}u^{a+6} + C$

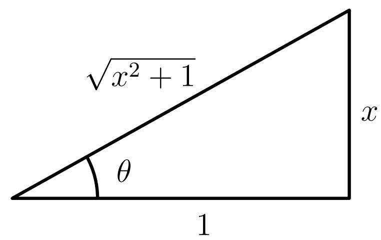
$\displaystyle = -\frac{1}{a+2}\cos^{a+2} x + \frac{2}{a+4}\cos^{a+4} x - \frac{1}{a+6}\cos^{a+6} x + C$

$\displaystyle = -\frac{1}{a+2}(\sqrt{1-x^2})^{a+2} + \frac{2}{a+4}(\sqrt{1-x^2})^{a+4} - \frac{1}{a+6}(\sqrt{1-x^2})^{a+6} + C$

3a. $\displaystyle\int \frac{x^3}{x^2+1}$

$$= \int \frac{\tan^3\theta}{\sec^2\theta}\ \sec^2\theta\,\mathrm{d}\theta$$

$$= \int \tan^3\theta\,\mathrm{d}\theta$$



$x = \tan\theta$

$\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$

$\sqrt{x^2+1} = \sec\theta$

3b. By polynomial long division, we get $\displaystyle\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$.

$$\int \frac{x^3}{x^2+1}\ \mathrm{d}x = \int x\,\mathrm{d}x - \int \frac{x}{x^2+1}\,\mathrm{d}x$$

The first integral is easy and the second needs a substitution with $u = x^2+1$ and $\mathrm{d}u = 2x\,\mathrm{d}x$.

So the integral is $\displaystyle\frac{1}{2}x^2 - \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}x^2 - \frac{1}{2}\ln|u| + C = \frac{1}{2}x^2 - \frac{1}{2}\ln|x^2+1| + C$.

4. $$\frac{1}{x^2(x-1)^3(x^2+16)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} + \frac{Fx+G}{x^2+16} + \frac{Hx+I}{(x^2+16)^2}$$

or $$\frac{1}{x^3(x-1)^2(x^2+4)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{Fx+G}{x^2+4} + \frac{Hx+I}{(x^2+4)^2}$$ or similar.

COMMENT: It can't hurt to write $\frac{A+Bx}{x^2}$ or $\frac{A+Bx+Cx^2}{x^3}$ here, but writing $\frac{C+Dx+Ex^2}{(x-1)^3}$ or $\frac{D+Ex}{(x-1)^2}$ was wrong because it was not useful for doing integrals.

5. $\displaystyle\frac{x^2+9x+3}{(x^2+1)(x+4)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+4}$.

Multiplying by $x+4$ and setting $x=-4$ gives us $\displaystyle\frac{16-36+3}{16+1} = C$, so $C=-1$.

Using $C=-1$ and multiplying through by the whole denominator:

$$x^2+9x+3 = (Ax+B)(x+4) + (-1)(x^2+1).$$

$$x^2+9x+3 = (A-1)x^2 + (4A+B)x + (4B-1), \text{ so}$$

$A-1=1 \qquad 4A+B=9 \qquad 4B-1=3,$ and we solve to get $A=2$ and $B=1$.

An alternative way to find A and B would be to multiply through by (x^2+1) and then set $x=i$:

$$\frac{i^2+9i+3}{(i+4)} = Ai+B+0 \qquad 2+9i = (Ai+B)(i+4) \qquad 2+9i = 4B-A+(4A+B)i$$

Then solve $4B-A=2,\quad 4A+B=9$.

$$\frac{x^2+9x+3}{(x^2+1)(x+4)} = \frac{2x+1}{x^2+1} - \frac{1}{x+4}.$$

5. $\dfrac{x^2+12x+14}{(x+1)^2(x+4)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+4}$.

Multiplying by $x+4$ and setting $x=-4$ gives us $\dfrac{16-48+14}{(-3)^2} = C$, so $C = \frac{-18}{9} = -2$.

Multiplying by $(x+1)^2$ and setting $x=-1$ gives us $\dfrac{1-12+14}{3} = B$, so $B=1$.

Using $B=1$ and $C=-2$ and multiplying through by the whole denominator:

$$x^2+12x+14 = A(x+1)(x+4) + (x+4) - 2(x+1)^2.$$

Comparing the coefficient of x^2 on both sides, we get $1 = A-2$, so $A=3$.

An alternative way to find A would be to multiply through by $(x+1)^2$, differentiate, and then set $x=-1$:

$$\frac{(-1+4)(2(-1)+12) - ((-1)^2+12(-1)+14)}{((-1)+4)^2} = A \qquad \text{so} \qquad A = \tfrac{27}{9} = 3.$$

$$\frac{x^2+12x+14}{(x+1)^2(x+4)} = \frac{3}{x+1} + \frac{1}{(x+1)^2} - \frac{2}{x+4}.$$

COMMENT: Setting up $\frac{x^2+12x+14}{(x+1)^2(x+4)} = \frac{Ax+B}{(x+1)^2} + \frac{C}{x+4}$ was not correct, because it was not useful for doing integrals! You would get $\frac{x^2+12x+14}{(x+1)^2(x+4)} = \frac{3x+4}{(x+1)^2} - \frac{2}{x+4}$. To integrate the first fraction, you would eventually have to compute the correct partial fractions!

5. $\dfrac{x^2+8x+25}{(x+1)(x+4)^2} = \dfrac{A}{x+1} + \dfrac{B}{x+4} + \dfrac{C}{(x+4)^2}$

Multiplying by $(x+4)^2$ and setting $x=-4$ gives us $\dfrac{16-32+25}{-3} = C$, so $C=-3$.

Multiplying by $x+1$ and setting $x=-1$ gives us $\dfrac{1-8+25}{(-3)^2} = A$, so $A=2$.

Using $A=2$ and $C=-3$ and multiplying through by the whole denominator:

$$x^2+8x+25 = 2(x+4)^2 + B(x+1)(x+4) - 3(x+1).$$

Comparing the coefficient of x^2 on both sides, we get $1 = 2+B$, so $B=-1$.

An alternative way to find B would be to multiply through by $(x+4)^2$, differentiate, and then set $x=-4$:

$$\frac{(-4+1)(2(-4)+8) - ((-4)^2+8(-4)+25)}{((-4)+1)^2} = B \qquad \text{so} \qquad B = \tfrac{-9}{9} = -1.$$

$$\frac{x^2+8x+25}{(x+1)(x+4)^2} = \frac{2}{x+1} - \frac{1}{x+4} - \frac{3}{(x+4)^2}$$

COMMENT: Setting up $\frac{x^2+8x+25}{(x+1)(x+4)^2} = \frac{A}{x+1} + \frac{Bx+C}{(x+4)^2}$ was not correct, because it was not useful for doing integrals! You would get $\frac{x^2+8x+25}{(x+1)(x+4)^2} = \frac{2}{x+1} + \frac{x+1}{(x+4)^2}$. To integrate the second fraction, you would eventually have to compute the correct partial fractions!

5. $\dfrac{x^2+x+10}{(x+1)(x^2+4)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+4}$

Multiplying by $x+1$ and setting $x=-1$ gives us $\dfrac{1-1+10}{5} = A$, so $A=2$.

Using $A=2$ and multiplying through by the whole denominator:

$$x^2+x+10 = 2(x^2+4) + (Bx+C)(x+1).$$

$$x^2+x+10 = (2+B)x^2 + (B+C)x + (8+C).$$

$2+B=1 \qquad B+C=1 \qquad 8+C=10,$ and we solve to get $B=-1$ and $C=2$.

An alternative way to find B and C would be to multiply through by (x^2+4) and then set $x=2i$:

$$\frac{(2i)^2+2i+10}{(2i+1)} = 0 + B(2i) + C \qquad 6+2i = (2Bi+C)(2i+1) \qquad 6+2i = C-4B+(2B+2C)i$$

Then solve $C-4B=6$, $\quad 2B+2C=2$.

$$\frac{x^2+x+10}{(x+1)(x^2+4)} = \frac{2}{x+1} + \frac{-x+2}{x^2+4}$$

6. Consider a thin sheet of material cut out by $y=\sqrt{1-x^2}$ and $y=0$ (a semicircle of radius 1.) Find the y-coordinate of the center of mass.

A vertical slice of thickness Δx at position x is a rectangle whose bottom is at $y=0$ and whose top is at $y=\sqrt{1-x^2}$. The mass of the slice is $\rho\cdot\sqrt{1-x^2}\Delta x$ (the area density times the area). The y-coordinate of the center of mass *of the slice* is in the middle, at height $\frac{1}{2}\sqrt{1-x^2}$. The mass-weighted integral of these y-coordinates is

$$\int_{-1}^{1} \rho\cdot\sqrt{1-x^2}\cdot\frac{1}{2}\sqrt{1-x^2}\,\mathrm{d}x = \frac{\rho}{2}\int_{-1}^{1}(1-x^2)\,\mathrm{d}x = \frac{\rho}{2}\left[x-\frac{1}{3}x^3\right]_{-1}^{1} = \frac{\rho}{2}\cdot\left(\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right) = \frac{2\rho}{3}.$$

The y-coordinate of the center of mass is this mass-weighted integral divided by the total mass:

$$\frac{2\rho}{3} \div \frac{\pi\rho}{2} = \frac{2\rho}{3}\cdot\frac{2}{\pi\rho} = \frac{4}{3\pi}.$$

7a. Since n appears to the 4th power in the denominator of the error estimate it, if we double n, we multiply the error by a factor of $\frac{1}{16}$. (You should have given a specific answer, which was $1\cdot\frac{1}{16}=\frac{1}{16}$ or $2\cdot\frac{1}{16}=\frac{1}{8}$ or $4\cdot\frac{1}{16}=\frac{1}{4}$ or $8\cdot\frac{1}{16}=\frac{1}{2}$.)

7b. In this case, Simpson's rule gives the exact value of the integral, which was either $\displaystyle\int_0^2 3x^2\,\mathrm{d}x = \left[x^3\right]_0^2 = 8$ or $\displaystyle\int_0^3 3x^2\,\mathrm{d}x = \left[x^3\right]_0^3 = 27$. There are two reasonable explanations:

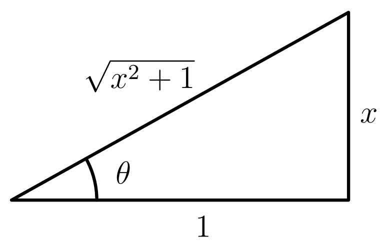
"Simpson's Rule approximates by fitting parabolas to the sample points. Since $3x^2$ is a parabola, the fitted parabola is exactly $y = 3x^2$, so we get exactly the correct answer."

"In the error estimate for Simpson's Rule, we can take $K = 0$, since the fourth derivative of $3x^2$ is 0. So the error estimate is zero, meaning that Simpson's Rule gives exactly the right answer."

CHALLENGE PROBLEM:

8. I wrote this problem because I was thinking about Problem 3. My idea was to do (basically) Problem 3a backwards, and then do Problem 3b. I still think this is a good way to do the problem. Here is my solution:

$$\int \tan^5\theta \, \mathrm{d}\theta$$
$$= \int x^5 \frac{1}{\sec^2\theta} \, \mathrm{d}x$$
$$= \int \frac{x^5}{x^2+1} \, \mathrm{d}x$$



$$x = \tan\theta$$
$$\mathrm{d}x = \sec^2\theta \, \mathrm{d}\theta$$
$$\mathrm{d}\theta = \tfrac{1}{\sec^2\theta} \, \mathrm{d}x$$
$$\sqrt{x^2+1} = \sec\theta$$

Polynomial long division lets us write $\displaystyle\int \frac{x^5}{x^2+1} \, \mathrm{d}x = \int \left(x^3 - x + \frac{x}{x^2+1}\right) \mathrm{d}x = \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2} \ln|x^2+1| + C.$

Thus $\displaystyle\int \tan^5\theta \, \mathrm{d}\theta = \frac{\tan^4\theta}{4} - \frac{\tan^2\theta}{2} + \frac{1}{2} \ln|\sec^2\theta| + C = \frac{\tan^4\theta}{4} - \frac{\tan^2\theta}{2} - \ln|\cos\theta| + C.$
(The last simplification used rules about the natural logarithm.)

It turns out, you could have done this problem using methods similar to problems we learned how to do: Substituting $\sec^2\theta - 1$ for $\tan^2\theta$ a few times, you could come up with the same answer as above, or with a different-looking—but equivalent—answer involving only secants.