MATH 241, Spring 2023, QUIZ 8 answers.

1. The sequences were different on different versions of the quiz. One of them was $a_n = (-1)^n \frac{n^2}{n^2+C}$ for some number C. No matter what C is, $\lim_{n\to\infty} \frac{n^2}{n^2+C}$ is 1, so as n goes to infinity, a_n is bouncing between numbers very close to 1 and very close to -1. This sequence diverges.

The other sequence was $a_n = D \cdot (\frac{1}{E})^n$ for some number D and some integer E greater than 1. The sequence $(\frac{1}{E})^n$ is geometric, and since E is greater than one, $\frac{1}{E}$ is positive and less than 1, so $(\frac{1}{E})^n$ converges to 0. No matter what D is $D \cdot (\frac{1}{E})^n$ also converges to 0.

2. Consider a sequence constructed as follows: The first term is $a_1 = 1$. Once I know a_{n-1}, I get a_n by moving $\frac{1}{n}$ of the distance up from a_{n-1} to the number 77. This sequence converges. Explain why we know it converges. (Be brief and don't write formulas or do calculations. For example, there is a very good answer that is 8 words. You could get away with 5 words.)

Here is the 8-word answer I had in mind: "It is increasing and bounded above by 77."

Acceptable 5-word answers: "It is monotonic and bounded." or "It is increasing and bounded."

My whole point in asking this question was to give you a sequence where all you could confidently say is that it is increasing and bounded. That way I could test that you know that "increasing and bounded" is enough to force convergence. From that point of view, I have nothing more to say about this problem. But I have a few more comments, because I think we have more to learn from this example. (Warning: If you read these comments, you might be led to think more deeply about sequences and convergence than you have before. But that might be a good thing.)
Besides the "increasing and bounded" answer, I didn't see any other answers on quizzes that I think are convincing. Some people said "Because its limit is 77". But the question is then "How do you know its limit is 77?" This problem is about explaining why! I wasn't even sure that the limit really was 77, but I checked on the computer, and it seems to be true. (How did I check? I asked the computer to find the first gazillion or so entries and saw that they were closing in on 77.) How could it not be 77? Well, at every step you move closer to 77, but maybe you move there so slowly that you don't ever get there. Sound implausible? OK, but it actually happens if you replace $\frac{1}{n}$ by $\frac{1}{n^2}$ in the desccription of the sequence. (In that case, the limit seems to be 39.)

Other people said "It converges because we keep adding things to it that are smaller and smaller" or maybe they said something similar, like "because $\lim_{n\to\infty} \frac{1}{n} = 0$". But remember the harmonic series example: It shows that if you keep adding things in, even if those things added in limit to zero, it is possible for your sum to be unbounded!

3. This problem concerned sequences defined by $a_n = \dfrac{a_{n-1} + a_{n-2}}{2}$. (You encountered sequences similar to these on your homework.) You don't know the sequence unless you are given a_1 and a_2, but once you knew them, you could (given enough time) compute any term in the sequence. Part a asked you to compute a_3 and a_4. Different versions of the quiz had different values of a_1 and a_2. These are the possibilities for your entries a_1, a_2, a_3, and a_4:

$8, \quad 4, \quad \frac{8+4}{2} = 6, \quad \frac{4+6}{2} = 5$

$10, \quad 2, \quad \frac{10+2}{2} = 6, \quad \frac{2+6}{2} = 4$

$7, \quad 3, \quad \frac{7+3}{2} = 5, \quad \frac{3+5}{2} = 4$

$9, \quad 1, \quad \frac{9+1}{2} = 5, \quad \frac{1+5}{2} = 3$

The sequence is not increasing (because $a_2 < a_1$). It is not decreasing (because $a_3 > a_2$). So it is not monotonic. (Literally, "not monotonic" means "not increasing and not decreasing".)

We could prove that the sequence converges and find the limit, but it would take some work. (That's why I decided not to ask part c for points—it's really beyond the scope of this class.) But the basic idea is that the distances between successive entries keeps getting divided by 2. And then, crucially, for any two consecutive entries, all remaining entries are between those two in value. That can be made into a proof using the ε-N definition of limit of a sequence. The limit is always $\frac{1}{3}$ of the first number plus $\frac{2}{3}$ of the second number.