MATH 241-003 and -004, Spring 2023, QUIZ 5 Answers.

1. $$\int_{-\infty}^{0} \frac{e^x}{1+e^x}\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^0 \frac{e^x}{1+e^x}\,\mathrm{d}x$$

(Substitution: $u = 1+e^x$, so $\mathrm{d}u = e^x\,\mathrm{d}x$)

$$= \lim_{a\to-\infty} \int_?^? \frac{1}{u}\,\mathrm{d}u = \lim_{a\to-\infty} \big[\ln|u|\big]_?^? = \lim_{a\to-\infty} \big[\ln|1+e^x|\big]_a^0 = \lim_{a\to-\infty} (\ln 2 - \ln|1+e^a|)$$

As $a \to -\infty$, $e^a \to 0$, so this limit is $\ln 2 - \ln 1 = \ln 2$.

Conclusion: $\displaystyle\int_{-\infty}^{0} \frac{e^x}{1+e^x}\,\mathrm{d}x = \ln 2.$

2. First, this is what you **can't do**:

$$\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x = \big[\ln|x|\big]_{-1}^{1} = 0 - 0 = 0.$$

This is wrong because there is a vertical asymptote at $x = 0$. Instead, break it into 2 integrals and do each one with a limit.

$$\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x = \int_{-1}^{0} \frac{1}{x}\,\mathrm{d}x + \int_{0}^{1} \frac{1}{x}\,\mathrm{d}x$$

$$= \lim_{b\to0^-} \big[\ln|x|\big]_{-1}^{b} + \lim_{a\to0^+} \big[\ln|x|\big]_{a}^{1}$$

$$= \lim_{b\to0^-} (\ln|b| - 0) + \lim_{a\to0^+} (0 - \ln|a|).$$

Both of these limits diverge, so $\displaystyle\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}x$ diverges.

You might think, "The first limit is ∞ and the second limit is $-\infty$, so they add up to 0." But that kind of arithmetic is problematic, because ∞ is not a number. There is infinite negative area below the curve and infinite positive area. That doesn't let us put any numerical value to the total area.