MATH 241, Spring 2023, QUIZ 10 answers.

If a function $f(x)$ has a power series representation (centered at 0), then the power series is the **Taylor Series** $\sum_{n=0}^{\infty} c_n x^n$, where $c_n = \dfrac{f^{(n)}(0)}{n!}$.

Both quiz questions are about the function $f(x) = \dfrac{1}{1+x^2}$.

1. Find the first few terms ($c_0 + c_1 x + c_2 x^2$) of the Taylor series (centered at 0) for $f(x)$.

$$\begin{aligned} f(x) &= \frac{1}{1+x^2} & \qquad c_0 &= f(0) &= 1 \\ f'(x) &= \frac{-2x}{(1+x^2)^2} & c_1 &= f'(0) &= 0 \\ f''(x) &= \frac{-2(1+x^2)^2 + 2x \cdot 4x(1+x^2)}{(1+x^2)^4} & c_2 &= \frac{f''(0)}{2!} &= -1 \end{aligned}$$

So the first few terms are $1 - x^2$.

2. Use a different technique to find the complete power series representation for $f(x)$.

Comparing $\dfrac{a}{1-r}$ with $\dfrac{1}{1+x^2}$, we see that $f(x)$ is a geometric series with $a = 1$ and $r = -x^2$. Thus

$$f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$