MA 241, Spring 2023, **POP** QUIZ B answers.

Problem 1

1. Consider the series $\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n x^n = 1 + \frac{1}{5}x + \frac{1}{25}x^2 + \frac{1}{125}x^3 + \cdots$.
   a. Find the radius of convergence of the series.
   b. Find the interval of convergence of the series.
   c. Give a formula for the function represented by this series on its interval of convergence.

Solution. Notice that this is a geometric series with $a = 1$ and $r = \frac{1}{5}x$. So it converges to $\frac{a}{1-r} = \frac{1}{1-\frac{1}{5}x}$ if $|\frac{1}{5}x| < 1$ (i.e. $|x| < 5$) and it diverges if $|\frac{1}{5}x| \geq 1$ (i.e. $|x| \geq 5$). That tells us the answers to all three parts:

a. The radius of convergence is 5.

b. The interval of convergence is $(-5, 5)$. (No need to test endpoints separately... since it's a geometric series, we already know it diverges at the endpoints.)

c. The function is $\frac{1}{1-\frac{1}{5}x} = \frac{5}{5-x}$.

Problem 2

2. Find the radius of convergence and the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

Solution. We'll use the Ratio Test to test convergence:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}x^{n+1}}{(n+1)!}}{\frac{2^n x^n}{n!}}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n}\right| = \lim_{n\to\infty} \left|\frac{2x}{(n+1)}\right| = 0$$

This is zero regardless of what x is, so the series converges for all x. That is, the radius of convergence is ∞. So the interval of convergence is $(-\infty, \infty)$. (No need to test endpoints separately... The interval is the whole number line, so there *are* no endpoints!)

By the way, we will see later that this series agrees with e^{2x} for all x. You should be able to do that problem using Taylor Series (for the final but not for Test 4).

Problem 3

3. Consider the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.
   a. Find the radius of convergence of the series.
   b. Find the interval of convergence of the series.

Solution. We'll again use the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}}\right| = \lim_{n\to\infty} \left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0$$

Again, this is zero regardless of what x is, so the series converges for all x. The radius of convergence is ∞. The interval of convergence is $(-\infty, \infty)$.

By the way, we showed in class that this series agrees with $\cos x$ for all x. You should be able to do that problem using Taylor Series.

Problem 4

4. Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.
   a. Find the radius of convergence of the series.
   b. Find the interval of convergence of the series.

Solution. Ratio Test again! (Sensing a pattern here? You're right.)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| = |x|$$

The Ratio Test says that this converges if that limiting ratio is < 1. That is the series converges if $|x| < 1$. So the radius of convergence is 1.
To find the exact interval of convergence, we need to test the endpoints ± 1.
Checking $x = 1$:
$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1^n}{n}$ converges by the Alternating Series Test (because $\frac{1}{n}$ decreases and limits to 0).
Checking $x = -1$:
$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}-\frac{1}{n}$ diverges. (It is -1 times the Harmonic Series).
So, the interval of convergence is $(-1, 1]$.