MA 241, Spring 2023, Nathan Reading
Synthesizing the Sequences/Series material (Sections 4.1–4.5 in the book)

Introduction. So far, the Sequences/Series chapter has been a long list of tools, and examples of how we use them. The task up until now has been for you to feel comfortable with why each tool works, and try using the tool.

Now, we need some synthesis. I hope this short summary helps you to "put it all together", but it's no substitute for reading the book or for looking back at lecture notes in detail.

A ***sequence*** is something like $a_1, a_2, a_3, \ldots$ (continuing infinitely). When we talk about a sequence, we want to know whether it ***converges*** to a ***limit*** or ***diverges*** (i.e. doesn't converge to a limit). If the sequence converges to L, we write $\lim_{n\to\infty} a_n = L$.

A ***series*** is the sum of a sequence: $\sum_{n=1}^{\infty} a_n$. That *means* the limit of the sequence of ***partial sums***. The n^{th} partial sum of the series is $S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$. The sum of the series is *defined* to be the limit of the sequence S_1, S_2, S_3, etc. If this sequence of partial sums converges to a limit S, then we say that $\sum_{n=1}^{\infty} a_n$ converges and we write $\sum_{n=1}^{\infty} a_n = S$. If the sequence of partial sums diverges, then we say $\sum_{n=1}^{\infty} a_n$ diverges. (So, even though almost all of the chapter is about series, we need to work with sequences in order to work with series.)

The questions we ask about a sequence or series are: **Does it converge or diverge? If it converges, can we find (or approximate) the limit?**

A series $\sum_{n=1}^{\infty} a_n$ ***converges absolutely*** if $\sum_{n=1}^{\infty} |a_n|$ converges. If the series converges absolutely, then it also converges. If it converges but doesn't converge absolutely, then we say it ***converges conditionally***. (Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.) So we might ask a more detailed question about a series: **Does the series converge absolutely, converge conditionally, or diverge?**

Working with sequences and series involves some basic understanding, a "toolbox", and a willingness/ability to try various tools until something works. (This means that you will benefit a lot from practice. For each of these tools, you should find the examples in the text and from lecture, and you should do extra problems from the book.)

The goal is to be given a sequence or series, not knowing in advance what tools to use, and find the answers to the questions mentioned above. So, let's talk toolbox:

Toolbox for sequences.

Treat them like functions. If there is a function $f(x)$ so that $a_n = f(n)$ for all integers $n \geq 1$, and if $\lim_{x\to\infty} f(x) = L$, then also $\lim_{n\to\infty} a_n = L$. But be careful! If $\lim_{x\to\infty} f(x)$ doesn't exist, $\lim_{n\to\infty} a_n$ might still exist. It would be good to have in your mind an example where $\lim_{x\to\infty} f(x)$ doesn't exist but $\lim_{n\to\infty} a_n$ does (for $a_n = f(n)$ as usual).

Add/subtract/multiply/divide them. Look at Theorem 2 on Page 12 of Chapter 4. But be careful! This only works if you know both sequences converge!

Squeeze Theorem. If your sequence is "between" two other sequences that go to the same limit, then your sequence also goes to that limit.

Monotone Convergence Theorem. An increasing sequence converges if and only if it is bounded above. A decreasing sequence converges if and only if it is bounded below.

Basic tools for series.

Test for Divergence. This is not the first thing in the text about series, but it is the first thing you should do when you see an unfamiliar series $\sum_{n=1}^{\infty} a_n$. Check whether $\lim_{n\to\infty} a_n = 0$. If not, the series diverges and you're done! If $\lim_{n\to\infty} a_n = 0$, then you still don't know if the series converges or diverges, and you need to think more.

Pull out constants, break up addition/subtraction. Another thing to think about early on when you meet an unfamiliar series: Can you pull out a constant? Are the terms sums or differences of other terms that I would know how to handle? These are Theorems 9 and 10 on pages 29 and 30 of Chapter 4. Loosely speaking, Theorem 9 is: If I have a *convergent* series and I multiply it by a constant, I get a convergent series and the limit is multiplied by that constant. If I have two *convergent* series and I add/subtract them term by term, I get a convergent series, and the limit is the sum/difference. Theorem 10 says that if I have a divergent series, then I can't make it convergent by multiplying by a constant or by adding/subtracting a *convergent* series.

Only the "tail" of the series matters. If you throw out a few terms at the beginning of the series, that won't change whether it converges or diverges (although it probably changes what the sum is, if it converges).

Fancier tools for series.

Geometric series. Given by the formula $\sum_{n=1}^{\infty} ar^{n-1}$. Or just think "The first term is a, and there is some number r such that at every step the next term is r times the previous term." Anyway, this converges to $\frac{a}{1-r}$ if $|r| < 1$. It diverges if $|r| \geq 1$. Since this is one of the few cases where we can actually find the sum of a series (with the tools we have in this course), you can expect to see it a lot.

The p-Series Test. This is pretty specific: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Telescoping series. This is the other case where we can actually find the sum of a series. The best description is in the book: "The partial sums simplify by cancellations in successive terms." There is no better explanation in words, so if you're having trouble seeing what "telescoping series" means, look at all the examples in the book and from lecture, and *think about what happens when we compute partial sums*. Sometimes, a little bit of manipulation is needed to see that the series is telescoping (as in the examples you have seen and read). Note that a telescoping series can be convergent or divergent.

The Alternating Series Test. ***Alternating*** means that the terms alternate sign: positive, negative, positive, negative, etc., or negative, positive, negative, positive, etc. Usually, but not always, this happens because the formula for the terms has $(-1)^n$ or $(-1)^{n+1}$ in it. If the absolute values of the terms are decreasing and limit to zero, the series converges. In this case, you also know that the n^{th} remainder R_n (defined to be $\sum_{i=1}^{\infty} a_i$ minus $\sum_{i=1}^{n} a_i$) has $|R_n| \leq |a_{n+1}|$.

The Integral Test. If you can write a function $f(x)$ that matches your series (i.e. $a_n = f(n)$ for integers $n \geq 1$) and if that function is continuous, decreasing, and positive-valued, then the series converges if and only if the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges. As a bonus, you get some information about what the series converges to: $\int_1^{\infty} f(x)\,\mathrm{d}x \;\leq\; \sum_{n=1}^{\infty} a_n \leq\; a_1 + \int_1^{\infty} f(x)\,\mathrm{d}x$.

The Comparison Test. If you have a convergent series where all terms are positive, and you change it so that all the terms get smaller (but stay positive!), the new series still converges. If you have a divergent series where all terms are positive and you change it so that all the terms get bigger, the new series still diverges.

The Limit Comparison Test. If two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have all positive terms and if $\lim_{n\to\infty} \frac{a_n}{b_n}$ is some number $K > 0$ (not infinity!), then the two series either both converge or both diverge.

The Ratio Test. This test about what the ratio $\frac{|a_{n+1}|}{|a_n|}$ does in the limit as $n \to \infty$. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges *absolutely*. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$ *or if this limit does not exist*, then the Ratio Test tells us nothing.

How to approach a series problem. Ask yourself:

- Do I get an easy win from the Test for Divergence?
- Can I pull apart constants or break apart addition or subtraction?
- Is my series geometric?
- Is it a p-series?
- Is it telescoping?
- Is it alternating?
- Is there a corresponding improper integral that I know how to compute? (Integral Test)
- Are my terms positive and less than the terms of a series that I know converges? Are my terms greater than the terms of a positive series that I know diverges? (Comparison Test)
- Do my terms, in the limit, approach being (a multiple of) the terms of a series that I already understand? (Limit Comparison Test)
- Does the ratio of absolute values of consecutive terms settle down to a limit? (Ratio Test)