

As usual, comments in italics are my explanations, which you didn't need to write on the test. Your versions of these problems may have been slightly different, and they may have been in a different order.

1. Find the following derivatives.

a. $\frac{d}{dx} 5^x = (\ln 5) \cdot 5^x$

b. $\frac{d}{dx} e^x = e^x$

c. $\frac{d}{dx} \ln x = \frac{1}{x}$

d. $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

2. Find the following derivatives.

a. $\frac{d}{dx} (e^{x^2+4} + \ln(7x) + 2) = 2xe^{x^2+4} - \frac{1}{7x} \cdot 7 = 2xe^{x^2+4} - \frac{1}{x}$. *Chain Rule*

b. $\frac{d}{dx} x^4 e^{-x} = x^4(-e^{-x}) + 4x^3 e^{-x} = (-x^4 + 4x^3)e^{-x}$. *Product Rule, using Chain Rule for $\frac{d}{dx} e^{-x}$*

c. $\frac{d}{dx} 8 \ln(\cos 3x) = 8 \frac{1}{\cos 3x} \cdot \frac{d}{dx} \cos 3x = 8 \frac{1}{\cos 3x} (-3 \sin 3x) = -24 \tan 3x$. *Chain Rule twice*

d. $\frac{d}{dx} (\arctan x)^8 = 8(\arctan x)^7 \frac{d}{dx} \arctan x = 8(\arctan x)^7 \frac{1}{1+x^2} = \frac{8(\arctan x)^7}{1+x^2}$. *Chain Rule*

e. To find $\frac{d}{dx} x^{\sin x}$, use logarithmic differentiation.

Take $\ln$ of both sides of $y = x^{\sin x}$:

$\ln y = \ln x^{\sin x}$ *Use logarithm rules!*

$\ln y = \sin x \cdot \ln x$ *Differentiate both sides $\frac{d}{dx}$*

$\frac{d}{dx} \ln y = \frac{d}{dx} (\sin x \cdot \ln x)$ *Use Chain rule on left and Product Rule on right.*

$\frac{1}{y} \frac{dy}{dx} = \sin x \cdot \frac{1}{x} + \cos x \cdot \ln x = \frac{\sin x}{x} + \cos x \cdot \ln x$ *Solve for $\frac{dy}{dx}$, remembering that $y = x^{\sin x}$.*

$\frac{dy}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \cos x \cdot \ln x \right)$

Answer: $\frac{d}{dx} x^{\sin x} = x^{\sin x} \left(\frac{\sin x}{x} + \cos x \cdot \ln x \right)$

3. Consider an object moving on a circle $x^2 + y^2 = 2$. When $x = -1$ and $y = 1$ and $\frac{dy}{dt} = 1$, what is $\frac{dx}{dt}$?

The point here is to differentiate both sides of the equation $x^2 + y^2 = 1$ with respect to t , and then use the given information. (Your given information might have been different, but the idea is the same.)

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$. *May as well solve for $\frac{dx}{dt}$:* $\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$

Then put in the given information: $\frac{dx}{dt} = -\frac{1}{-1} \cdot 1 = 1$.

4. Consider the curve given by $xy = x^2y^5$.

a. Use implicit differentiation to find a formula for $\frac{dy}{dx}$.

Take $\frac{d}{dx}$ of both sides: $x \cdot \frac{dy}{dx} + 1 \cdot y = x^2 \cdot 5y^4 \frac{dy}{dx} + 2x \cdot y^5$

Solve for $\frac{dy}{dx}$: $\frac{dy}{dx}(x - 5x^2y^4) = 2xy^5 - y$ $\frac{dy}{dx} = \frac{2xy^5 - y}{x - 5x^2y^4}$

b. Verify that the point $(1, 1)$ is on the curve.

$$1 \cdot 1 - 1^2 \cdot 1^5 = 1 - 1 = 0.$$

c. Find the slope of the tangent line to the curve at the point $(1, 1)$.

At $(x, y) = (1, 1)$: $\frac{dy}{dx} = \frac{2 \cdot 1 \cdot 1^5 - 1}{1 - 5 \cdot 1^2 \cdot 1^4} = \frac{2 - 1}{1 - 5} = -\frac{1}{4}$

5. (ONE VERSION:) Analyze the function $f(x) = x^3 + 3x^2$. Specifically:

a. Where are there local maxima/minima or inflection points?

$$f'(x) = 3x^2 + 6x = 3x(x + 2), \quad \text{never undefined, zero at } x = 0 \text{ and } x = -2.$$

$$f'(x) > 0 \text{ when } x > 0 \text{ or } x < -2 \text{ and } f'(x) < 0 \text{ for } -2 < x < 0.$$

So the First Derivative Test says:

There is a local minimum at $x = 0$.

There is a local maximum at $x = -2$.

$$f''(x) = 6x + 6 = 6(x + 1), \quad \text{zero at } x = -1.$$

$$f''(x) > 0 \text{ when } x > -1, \text{ so } f(x) \text{ is concave up for } x > -1.$$

$$f''(x) < 0 \text{ for } x < -1, \text{ so } f(x) \text{ is concave down for } x < -1.$$

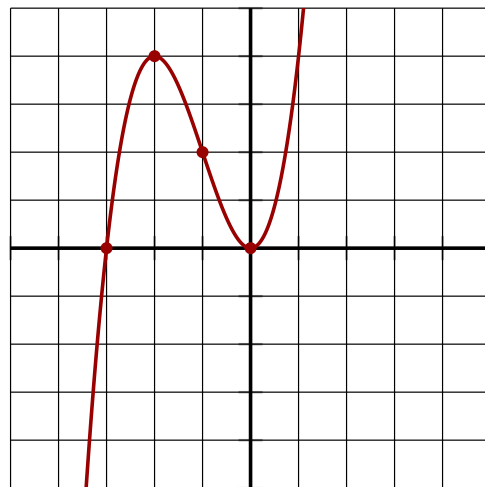
There is an inflection point at $x = -1$.

(Also $f''(0) > 0$ and $f''(-2) < 0$, so the Second Derivative Test was another way to see that there is a local minimum at $x = 0$ and a local maximum at $x = -2$.)

b. Draw a reasonable sketch of the graph of $f(x)$. Make the sketch accurate in the sense of where $f(x)$ is increasing/decreasing and in the sense of concavity. Mark (as dots) all local maxima and/or local minima, all inflection points, and all x -intercepts and y -intercepts.

The only thing we need to compute is x -intercepts, i.e. where $f(x) = 0$: $x^3 + 3x^2 = x^2(x + 3) = 0$ when $x = 0$ or $x = -3$.

Yes, this is a computer generated graph of the function, but you should make sure you are clear how to make a sketch that is accurate



5. (ANOTHER VERSION:) Analyze the function $f(x) = x^3 - 3x^2$. Specifically:

a. Where are there local maxima/minima or inflection points?

$$f'(x) = 3x^2 - 6x = 3x(x - 2), \quad \text{never undefined, zero at } x = 0 \text{ and } x = 2.$$

$$f'(x) > 0 \text{ when } x > 2 \text{ or } x < 0 \text{ and } f'(x) < 0 \text{ for } 0 < x < 2.$$

So the First Derivative Test says:

There is a local minimum at $x = 2$.

There is a local maximum at $x = 0$.

$$f''(x) = 6x - 6 = 6(x - 1), \quad \text{zero at } x = 1.$$

$$f''(x) > 0 \text{ when } x > 1, \text{ so } f(x) \text{ is concave up for } x > 1.$$

$$f''(x) < 0 \text{ for } x < 1, \text{ so } f(x) \text{ is concave down for } x < 1.$$

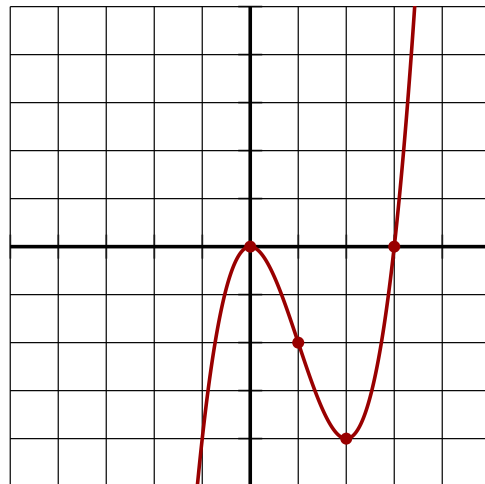
There is an inflection point at $x = 1$.

(Also $f''(0) < 0$ and $f''(-2) > 0$, so the Second Derivative Test was another way to see that there is a local minimum at $x = 2$ and a local maximum at $x = 0$.)

b. Draw a reasonable sketch of the graph of $f(x)$. Make the sketch accurate in the sense of where $f(x)$ is increasing/decreasing and in the sense of concavity. Mark (as dots) all local maxima and/or local minima, all inflection points, and all x -intercepts and y -intercepts.

The only thing we need to compute is x -intercepts, i.e. where $f(x) = 0$: $x^3 - 3x^2 = x^2(x - 3) = 0$ when $x = 0$ or $x = 3$.

Yes, this is a computer generated graph of the function, but you should make sure you are clear how to make a sketch that is accurate



6. Find the linearization $L_1(x)$ of the function $\sqrt[5]{x}$ at $x = 1$ and use it to approximate $\sqrt[5]{0.9}$, $\sqrt[5]{0.95}$, $\sqrt[5]{1.05}$, or $\sqrt[5]{1.1}$.

Set $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$ and compute $f'(x) = \frac{1}{5}x^{-\frac{4}{5}}$, so $f(1) = \sqrt[5]{1} = 1$ and $f'(1) = \frac{1}{5}$.

$L_1(x) = 1 + \frac{1}{5}(x - 1)$. (In general, $L_c(x) = f(c) + f'(c)(x - c)$, and here $c = 1$.)

So we approximate $\sqrt[5]{x} \approx 1 + \frac{1}{5}(x - 1)$.

$$\sqrt[5]{0.9} \approx 1 + \frac{1}{5}(0.9 - 1) = 1 + \frac{1}{5}(-0.1) = 1 - 0.02 = 0.98$$

$$\sqrt[5]{0.95} \approx 1 + \frac{1}{5}(0.95 - 1) = 1 + \frac{1}{5}(-0.05) = 1 - 0.01 = 0.99$$

$$\sqrt[5]{1.05} \approx 1 + \frac{1}{5}(1.05 - 1) = 1 + \frac{1}{5}(0.05) = 1 + 0.01 = 1.01$$

$$\sqrt[5]{1.1} \approx 1 + \frac{1}{5}(1.1 - 1) = 1 + \frac{1}{5}(0.1) = 1 + 0.02 = 1.02$$

7. Find the absolute maximum and minimum of $f(x) = \frac{2x}{x^2 + 1}$ on the interval $[0, 3]$.

$f'(x) = \frac{(x^2 + 1) \cdot 2 - 2x \cdot 2x}{(x^2 + 1)^2} = \frac{2 - 2x^2}{(x^2 + 1)^2}$. This is zero when the numerator is zero, so $x = \pm 1$, and this is never undefined because the denominator is never zero. Only one of these critical points ($x = 1$) is in the interval $[0, 3]$. We check the critical point and the endpoints:

$$f(0) = \frac{2 \cdot 0}{0^2 + 1} = 0.$$

$$f(1) = \frac{2 \cdot 1}{1^2 + 1} = \frac{2}{2} = 1$$

$$f(3) = \frac{2 \cdot 3}{3^2 + 1} = \frac{6}{10} = \frac{3}{5}$$

Absolute maximum is 1, attained at $x = 1$.

Absolute minimum is 0, attained at $x = 0$.

8. This problem is about the same function $f(x) = \frac{2x}{x^2 + 1}$ as in Problem 7, and you can reuse much of your work from there.

Compute $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow -\infty} f(x) = 0$ and $f(-1) = \frac{2 \cdot (-1)}{(-1)^2 + 1} = \frac{-2}{2} = -1$.

Absolute minimum of $f(x)$ on $[0, \infty)$ is 0, because 0 is the smallest value at a critical point or endpoint (and is the same as one of the limits).

Absolute minimum of $f(x)$ on $[3, \infty)$ does not exist, because the limit of $f(x)$ as $x \rightarrow \infty$ is less than the value of $f(x)$ at the other endpoint, and there are no critical points in the interval.

Absolute minimum of $f(x)$ on $(-\infty, \infty)$ is -1 because the value $f(-1) = -1$ is less than the value at the other critical point and also less than the limits at the endpoints.

9. This problem concerns a general “Exponentiation Rule.”

a. Use logarithmic differentiation to find a formula for $\frac{d}{dx} f(x)^{g(x)}$. (You can assume that f and g are differentiable and that $f(x)$ is positive, so that all the manipulations you do really make sense.)

This was just like Problem 2e. You just had to not panic about “ $f(x)$ ” and “ $g(x)$.”

Take $\frac{d}{dx} \ln$ of both sides of $y = f(x)^{g(x)}$:

$$\frac{d}{dx} \ln y = \frac{d}{dx} \ln f(x)^{g(x)}$$

$$\frac{d}{dx} \ln y = \frac{d}{dx} g(x) \ln f(x)$$

$$\frac{1}{y} \frac{dy}{dx} = g(x) \cdot \frac{1}{f(x)} \cdot f'(x) + g'(x) \cdot \ln f(x) = g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x)$$

$$\frac{dy}{dx} = f(x)^{g(x)} \left(g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x) \right)$$

Answer: $\boxed{\frac{d}{dx} f(x)^{g(x)} = f(x)^{g(x)} \left(g(x) \frac{f'(x)}{f(x)} + g'(x) \ln f(x) \right)}$

b. Show how your formula proves the usual Power Rule.

This is the case where $f(x) = x$ and $g(x)$ is the constant p . The formula says:

$$\frac{d}{dx} x^p = x^p \left(p \frac{1}{x} + 0 \ln x \right), \quad \text{which simplifies to } \frac{d}{dx} x^p = p x^{p-1}.$$

c. Show how your formula proves that $\frac{d}{dx} e^x = e^x$.

This is the case where $f(x) = e$ and $g(x) = x$. The formula says:

$$\frac{d}{dx} e^x = e^x \left(x \frac{0}{e} + 1 \ln e \right), \quad \text{which simplifies to } \frac{d}{dx} e^x = e^x.$$

10. (2 points) Suppose $f(x)$ is a one-to-one differentiable function, so that it has an inverse function $f^{-1}(x)$ and a derivative $f'(x)$. Write an expression for $\frac{d}{dx} f^{-1}(x)$ in terms of $f^{-1}(x)$ and $f'(x)$.

This problem asks you to understand the process we used to find the derivatives of $\arcsin x$, $\arctan x$, $\ln x$, and $\log_a x$ in a very general setting. What was the process? Set $y = f^{-1}(x)$ (for convenience) and then do implicit differentiation.

$$y = f^{-1}(x)$$

$$f(y) = x$$

$$f'(y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{f'(y)}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}}$$

When $f(x)$ was $\sin x$, $\tan x$, e^x , or a^x , we used properties of the function to simplify this.