MATH 141, Fall 2025, TEST 1 Answers and comments

Italics are my comments and ordinary typeface is what you might/should have answered.

1. *Your test had fewer questions than this, but all your answers are on this list, except you might have had a different constant (instead of 3) for the last two.*

$f(-7) = -1$.	$\lim_{x\to -7} f(x)$ does not exist.	$\lim_{x\to -7^+} f(x) = -5$.	$\lim_{x\to -7^-} f(x) = 4$.
$f(-5) = 4$.	$\lim_{x\to -5} f(x) = 1$.	$\lim_{x\to -5^+} f(x) = 1$.	$\lim_{x\to -5^-} f(x) = 1$.
$f(0) = 0$.	$\lim_{x\to 0} f(x)$ does not exist.	$\lim_{x\to 0^+} f(x) = -1$.	$\lim_{x\to 0^-} f(x) = 1$.
$f(-9) = -2$.	$\lim_{x\to -9} f(x) = 3$.	$\lim_{x\to -9^+} f(x) = 3$.	$\lim_{x\to -9^-} f(x) = 3$.
$f(6)$ is undefined.	$\lim_{x\to 6} f(x) = -\infty$.	$\lim_{x\to 6^+} f(x) = -\infty$.	$\lim_{x\to 6^-} f(x) = -\infty$.
$3f(2) = 9$.	$\lim_{x\to 2^-} 3f(x) = 9$.	$\lim_{x\to 4} f(x)$ does not exist	

2a. or 2c. $\lim_{x\to 1} \dfrac{1}{(x-1)^2} = \infty$

As x gets close to 1, from above or below, $(x-1)^2$ gets small positive, so $\frac{1}{(x-1)^2}$ gets large positive.

2b. $\lim_{x\to\infty} \dfrac{12x^7 + x^6 - 80x^4 + 24x - 3}{ax^7 + 15x^5} = \dfrac{12}{a}$ *Your test had some specific number a.*

As x gets large positive, the $12x^7$ term on top gets large much faster than the other terms on top. The ax^7 term on bottom gets large fastest. So we're taking the limit $\lim_{x\to\infty} \dfrac{12x^7}{ax^7} = \lim_{x\to\infty} \dfrac{12}{a} = \dfrac{12}{a}$.

If you want to do 2b more formally like the book would:

$$\lim_{x\to\infty} \frac{12x^7 + x^6 - 80x^4 + 24x - 3}{ax^7 + 15x^5} = \lim_{x\to\infty} \frac{12x^7 + x^6 - 80x^4 + 24x - 3}{ax^7 + 15x^5} \cdot \frac{\frac{1}{x^7}}{\frac{1}{x^7}}$$
$$= \lim_{x\to\infty} \frac{12 + \frac{1}{x} - \frac{80}{x^3} + \frac{24}{x^6} - \frac{3}{x^7}}{a + \frac{15}{x^2}} = \frac{\lim_{x\to\infty}(12 + \frac{1}{x} - \frac{80}{x^3} + \frac{24}{x^6} - \frac{3}{x^7})}{\lim_{x\to\infty}(a + \frac{15}{x^2})} = \frac{12}{a}.$$

2c. or 2a. *Your test had* $\lim_{x\to 0} \dfrac{(x-a)(x-1)}{x-1}$ *for some value of a, but the numerator wasn't factored. The point of this problem is that $f(1)$ is undefined because it gives division by zero in the formula. But when you factor the numerator and cancel the $x-1$, you see that this is the function $x-a$, with a removable discontinuity at $x=1$. The limit as $x \to 1$ is $1-a$. So you might have written something like:*

$$\lim_{x\to 1} \frac{x^2 - 6x + 5}{x-1} = \lim_{x\to 1} \frac{(x-5)(x-1)}{x-1} = \lim_{x\to 1}(x-5) = 1 - 5 = -4.$$

3. *Different tests had different values of a.*

a. $\lim_{x\to 3} f(x) = a$.

b. $\lim_{x\to 3^-} f(x) = a$.

Since f is continuous, $\lim_{x\to 3} f(x) = f(3) = a$. That also means that both one-sided limits equal the two-sided limit, so $\lim_{x\to 3^-} f(x) = f(3) = a$.

4. or 5. $f(x) = x^3 - 5x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - 5(x+h) - (x^3 - 5x)}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - 5x - 5h - x^3 + 5x}{h} \\ &= \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3 - 5h}{h} \\ &= \lim_{h\to 0} (3x^2 + 3hx + h^2 - 5) \\ &= 3x^2 - 5 \end{aligned}$$

Your function was $f(x) = x^3 - ax$ for some number a, but your computation would look the same.

5 or 4. $f(x) = \dfrac{1}{x}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \left[\frac{1}{h} \cdot \left(\frac{1}{x+h} - \frac{1}{x} \right) \right] \\ &= \lim_{h\to 0} \left(\frac{1}{h} \cdot \frac{x - (x+h)}{(x+h)x} \right) \\ &= \lim_{h\to 0} \left(\frac{1}{h} \cdot \frac{-h}{(x+h)x} \right) \\ &= \lim_{h\to 0} \frac{-1}{(x+h)x} \\ &= \frac{-1}{x^2} \end{aligned}$$

6. *There were different versions of some of these problems and they might have been in a different order.*

a. $\dfrac{\mathrm{d}}{\mathrm{d}x}\, x^6 = 6x^5$ or a similar problem.

b. $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \sqrt[3]{x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\, x^{\frac{1}{3}} = \dfrac{1}{3} x^{-\frac{2}{3}} = \dfrac{1}{3\sqrt[3]{x^2}}$ or a similar problem.

c. $\dfrac{\mathrm{d}}{\mathrm{d}x}\, 9 = 0$ or a similar problem.

d. $\dfrac{\mathrm{d}}{\mathrm{d}x}\, x^{\frac{3}{4}} = \dfrac{3}{4} x^{-\frac{1}{4}}$ or a similar problem.

e. $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \dfrac{1}{x^5} = \dfrac{\mathrm{d}}{\mathrm{d}x}\, x^{-5} = -5x^{-6} = \dfrac{-5}{x^6}$ or a similar problem.

f. $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \sin x = \cos x$

g. $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \cos x = -\sin x$

h. $\dfrac{\mathrm{d}}{\mathrm{d}x}\, \tan x = \dfrac{\mathrm{d}}{\mathrm{d}x}\, \dfrac{\sin x}{\cos x} = \dfrac{(\cos x)\cdot(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$

7. *There were different versions of some of these problems and they were in different orders. I'll do them all here:*

a. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(3x^5 - 3 + \cos x\right) = 15x^4 - \sin x$

b. $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin\left(x + \dfrac{1}{x}\right) = \cos\left(x + \dfrac{1}{x}\right) \cdot \left(1 - \dfrac{1}{x^2}\right)$

*It is **wrong** if you write* $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin\left(x + \dfrac{1}{x}\right) = \cos\left(x + \dfrac{1}{x}\right) \cdot 1 - \dfrac{1}{x^2}$. *It's just not the same function.*

c. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2-3}{2x-1} = \dfrac{(2x-1)2x - (x^2-3)\cdot 2}{(2x-1)^2} = \dfrac{2x^2-2x+6}{(2x-1)^2}$

c. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2-2}{3x-1} = \dfrac{(3x-1)2x - (x^2-2)\cdot 3}{(3x-1)^2} = \dfrac{3x^2-2x+6}{(3x-1)^2}$

c. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{2x^2-3}{x-1} = \dfrac{(x-1)4x - (2x^2-3)\cdot 1}{(x-1)^2} = \dfrac{2x^2-4x+3}{(x-1)^2}$

c. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^2-3}{x-2} = \dfrac{(x-2)2x - (x^2-3)\cdot 1}{(x-2)^2} = \dfrac{x^2-4x+3}{(x-2)^2}$

d. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\sin x}{x} = \dfrac{x\cos x - \sin x}{x^2}$

d. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\cos x}{x} = \dfrac{-x\sin x - \cos x}{x^2}$

d. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x}{\sin x} = \dfrac{\sin x - x\cos x}{\sin^2 x}$

d. $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x}{\cos x} = \dfrac{\cos x + x\sin x}{\cos^2 x}$

e. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^7\cos x\right) = 7x^6\cos x - x^7\sin x$

f. $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos^3(4x) = 3\cos^2(4x)\cdot\dfrac{\mathrm{d}}{\mathrm{d}x}\cos(4x) = 3\cos^2(4x)\cdot(-4\sin(4x)) = -12\cos^2(4x)\sin(4x)$

8a. If f is differentiable at x_0, then

$$\begin{aligned}
f^{\text{fun}}(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h} \\
&= \lim_{h\to 0} \frac{\big(f(x_0+h)-f(x_0)\big)-\big(f(x_0-h)-f(x_0)\big)}{2h} \\
&= \lim_{h\to 0} \left(\frac{f(x_0+h)-f(x_0)}{2h} - \frac{f(x_0-h)-f(x_0)}{2h}\right) \\
&= \lim_{h\to 0} \left(\frac{1}{2}\cdot\frac{f(x_0+h)-f(x_0)}{h} - \frac{1}{2}\cdot\frac{f(x_0-h)-f(x_0)}{h}\right) \\
&= \lim_{h\to 0} \left(\frac{1}{2}\cdot\frac{f(x_0+h)-f(x_0)}{h} + \frac{1}{2}\cdot\frac{f(x_0-h)-f(x_0)}{-h}\right) \\
&= \frac{1}{2}\lim_{h\to 0} \left(\frac{f(x_0+h)-f(x_0)}{h}\right) + \frac{1}{2}\lim_{h\to 0} \left(\frac{f(x_0-h)-f(x_0)}{-h}\right)
\end{aligned}$$

The first term here is $\frac{1}{2}f'(x_0)$. But the second term is also $\frac{1}{2}f'(x_0)$. Why? Let $k=-h$. Then as $h\to 0$, also $k\to 0$, so the second term is $\frac{1}{2}\lim_{k\to 0}\left(\frac{f(x_0+k)-f(x_0)}{k}\right)$, which is $\frac{1}{2}f'(x_0)$. So $f^{\text{fun}}(x_0) = \frac{1}{2}f'(x_0)+\frac{1}{2}f'(x_0) = f'(x_0)$.

8b. There are lots of possibilities. Here is one: Take $f(x)=\frac{x^2}{x}$, so that $f(x)=x$ everywhere except at $x=0$, where it is undefined. Then

$$f^{\text{fun}}(0) = \lim_{h\to 0}\frac{f(0+h)-f(0-h)}{2h} = \lim_{h\to 0}\frac{(0+h)-(0-h)}{2h} = \lim_{h\to 0} 1 = 1.$$

8c. Again, there are lots of possibilities. Here is one: Take $f(x)=|x|$, which has a "corner" at 0 and so is not differentiable there. But

$$f^{\text{fun}}(0) = \lim_{h\to 0}\frac{f(0+h)-f(0-h)}{2h} = \lim_{h\to 0}\frac{|0+h|-|0-h|}{2h} = \lim_{h\to 0} 0 = 0.$$

9. *There are lots of possible correct answers. I'll just give one for each case.*

a. $\lim_{x\to\infty} f(x)g(x)=0$: $\quad f(x)=0 \quad g(x)=x$

b. $\lim_{x\to\infty} f(x)g(x)=\infty$: $\quad f(x)=\dfrac{1}{x} \quad g(x)=x^2$

c. $\lim_{x\to\infty} f(x)g(x)=-\infty$: $\quad f(x)=-\dfrac{1}{x} \quad g(x)=x^2$

d. $\lim_{x\to\infty} f(x)g(x)=3$: $\quad f(x)=\dfrac{3}{x} \quad g(x)=x$

e. $\lim_{x\to\infty} f(x)g(x)$ does not exist: $\quad f(x)=\dfrac{\sin x}{x} \quad g(x)=x$