

How close does the curve $y = x^2$ get to the point $(0, \frac{3}{2})$, and where on the curve is this minimum distance attained?

The distance from a point (x, y) to $(0, \frac{3}{2})$ is $\sqrt{(x-0)^2 + (y-\frac{3}{2})^2}$. If the point is on the curve $y = x^2$, then this is $\sqrt{(x-0)^2 + (x^2-\frac{3}{2})^2}$, which equals $\sqrt{x^4 - 2x^2 + \frac{9}{4}}$. Finding where that distance is as small as possible is the same as finding where $x^4 - 2x^2 + \frac{9}{4}$ is as small as possible. We'll do the easier thing (but see a comment below) and find the absolute minimum of $f(x) = x^4 - 2x^2 + \frac{9}{4}$:

$f'(x) = 4x^3 - 4x$. Solve $4x^3 - 4x = 0$: $4x(x+1)(x-1) = 0$, so $x = -1, 0$, or 1 .

Since $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ are both ∞ , the absolute minimum is attained at one of these critical points. Calculate $f(-1) = \frac{5}{4}$, $f(0) = \frac{9}{4}$, $f(1) = \frac{5}{4}$. So the minimum is attained at $x = \pm 1$, where $f(x) = \frac{5}{4}$, so the distance is $\frac{\sqrt{5}}{2}$. Both of these points have $y = 1$.

Answer: The closest the curve gets is $\frac{\sqrt{5}}{2}$, and that distance is attained at the points $(-1, 1)$ and $(1, 1)$.

Comments:

- You could have (and probably should have) been less wordy than what I wrote above, but some words are helpful.
- You didn't have to use the trick where you ignored the square root to find the critical points. Let's see how it comes out if you leave the square root in:

$$f(x) = \sqrt{x^4 - 2x^2 + \frac{9}{4}}$$

$$f'(x) = \frac{1}{2}(x^4 - 2x^2 + \frac{9}{4})^{-\frac{1}{2}} \cdot (4x^3 - 4x) = \frac{4x^3 - 4x}{\sqrt{x^4 - 2x^2 + \frac{9}{4}}}$$

This is never undefined. (Its denominator is the distance from a point on the curve to $(0, \frac{3}{2})$, and that's never 0, because $(0, \frac{3}{2})$ is not on the curve.

This is zero when its denominator is zero, so we're still solving $4x^3 - 4x = 0$.

- A sneakier way to do it was to write the distance as a function of y , again using the "ignore the square root" trick. This only makes sense on the interval $[0, \infty)$, because $y = x^2$, so we have to find the maximum on $[0, \infty)$.

$$f(y) = y^2 - 2y + \frac{9}{4}.$$

$f'(y) = 2y - 2$, which is zero when $y = 1$.

Since $\lim_{y \rightarrow \infty} f(y) = \infty$, the minimum is either the endpoint $y = 0$ or the critical point $y = 1$.

Check: $f(0) = \frac{9}{4}$ and $f(1) = \frac{5}{4}$, so the minimum is attained at $y = 1$.

But there are two x -values that give $y = 1$.

Answer: The closest the curve gets is $\frac{\sqrt{5}}{2}$, and that distance is attained at the points $(-1, 1)$ and $(1, 1)$.

Be careful: The sneaky way may not work for other problems that look a lot like this.