MA 724
Homework 5, Comments and some solutions.

Problem 9

Applying Theorem 2.7, we know that $F^\diamond$ is a polytope of dimension $\dim(P) - \dim(F) - 1$. We can't just blindly apply the geometric operation of polar duality, because F typically is not full dimensional and does not contain the origin in its interior. But we can get something combinatorially dual to $F^\diamond$ by restricting to the affine hull of $F^\diamond$, then translating until $\vec{0}$ is in the interior of $F^\diamond$, and *then* doing polar duality. (If we don't restrict to the affine hull, then the interior of $F^\diamond$ can be empty, and then things go wrong because we need $\vec{0}$ in the interior, not just the relative interior.) This construction is what Ziegler meant (and indeed, he was thinking of the $^\triangle$ symbol as exactly this combinatorial duality).

Now P/F, defined in this way, is again a polytope of dimension $\dim(P) - \dim(F) - 1$. The face lattice of $F^\diamond$ is isomorphic to the interval $[P^\diamond, F^\diamond]$ in $L(P^\triangle)$. The face lattice of P/F is dual to the face lattice of $F^\diamond$, so it is isomorphic to the interval $[F, P]$ in $L(P)$.

To obtain P/F as an iterated face figure, it is convenient to fix, first, a saturated chain $\emptyset = F_0 < F_1 < \cdots < F_k = F$. Then P/F_1 is a vertex figure and $L(P/F_1)$ is isomorphic to $[F_1, P]$. There is a saturated chain corresponding to $F_1 < \cdots < F_k = F$ in $L(P/F_1)$, and we can repeat, forming $(P/F_1)/F_2$, then $((P/F_1)/F_2)/F_3$, and so forth until we reach a polytope whose face lattice is isomorphic to $[F, P]$. This is P/F.

We'll sketch a more direct construction of P/F. (There is another construction below, which you might like better.) We'll first construct what I'll call the *local polyhedron* $P|_F$ of P at F. This is the polyhedron defined by all of the facet-defining inequalities for facets of P containing F. If $F = \emptyset$, then $P|_F = P$, so assume that $F \neq \emptyset$. There is a bijection η between faces of P containing F and nonempty faces of $P|_F$, with inverse θ defined as follows: Given a face G of P containing F, η maps G to the intersection of the facets of $P|_F$ containing G. Given a nonempty face E of $P|_F$, θ maps E to $P \cap E$.

If G is a face of P containing F, and U is the set of facets of $P|_F$ containing G, then $\theta(\eta(G)) = P \cap \bigcap_{F' \in U} F' = \bigcap F' \in U(P \cap F')$. The latter is the intersection of all of the facets of P containing G, so it equals G. If E is a nonempty face of $P|_F$, then $\eta(\theta(E))$ is the intersection of all of the facets of $P|_F$ containing $P \cap E$. But a facet of $P|_F$ contains $P \cap E$ if and only if it contains E, so $\eta(\theta)$ is the intersection of all of the facets of $P|_F$ containing G, which equals G.

Now, to make a polytope with the right face lattice, mod out by the lineality space of $P|_F$. This gives a polyhedron with a unique vertex (corresponding to the minimal nonempty face $\eta(F)$ of $P|_F$). Take the vertex figure of that vertex. The result is a polytope isomorphic to P/F.

Here is another direct construction that you might like better. Choose any point $\vec{x}$ in the relative interior of F, and let S be an affine subspace containing $\vec{x}$ and complementary to $\operatorname{aff}(F)$ (meaning that the intersection of the two is $\vec{x}$ and their dimensions sum to d). The intersection $P \cap S$ is a polytope of dimension $\dim(P) - \dim(F)$, and $\vec{x}$ is a vertex of $P \cap S$. Then R/F is the vertex figure $(P \cap S)/\vec{x}$.

Here is a sketch of why this works. Since we can obtain P/F as an iterated vertex figure, we are basically intersecting P with a hyperplane, then another, etc. In all, we intersect P with $\dim(F) + 1$ hyperplanes. This amounts to intersecting P with a subspace of dimension $\dim(P) - \dim(F) - 1$. (If the hyperplanes are not in independent directions, we would end up with a polytope of the wrong dimension.) The construction above intersects P with a subspace of dimension $\dim(P) - \dim(F)$ and then intersects the result with another hyperplane, so in all it amounts to intersecting P with a subspace of dimension $\dim(P) - \dim(F) - 1$. The detail that needs to be checked is that we can choose the complementary subspace and construct all the vertex figures so that these two subspaces of dimension $\dim(P) - \dim(F) - 1$ actually coincide. I'll leave that to you if you are interested, but the point is that we have a *lot* of freedom in constructing these subspaces.

Problem 10

Take $\vec{y}$ to be the center of a square in $\mathbb{R}^2$, and you'll find a counterexample to both statements (iii) and (iv) if the word "point" is replaced by "vertex" in both statements. A counterexample just for (iv) is easier: A triangle has more than one interior point!

But wait! Why doesn't this counterexample to (iii) contradict Caratheodory's Theorem? (Look carefully at the theorem.)

Lecture 2, Problem 14

Part (i). First, here is a true statement for polyhedra.

Proposition 1. *Suppose $P(A,\vec{z})$ is a nonempty polyhedron. An inequality that appears as a row of $A\vec{x} \le \vec{z}$ is redundant if and only if it can be written as a positive combination of other inequalities in the system and the inequality $\mathbb{0}\vec{x} \le 1$.*

Proof. The "if" direction is trivial. Suppose the inequality $\vec{a}_0\vec{x} \le z_0$ is redundant, and let $A'\vec{x} \le \vec{z}\,'$ be the system obtained by deleting that row. Since $P(A,\vec{z}) \neq \emptyset$, condition (ii) of Farkas III does not hold (by Farkas I or an easy proof). Thus Farkas III says that there exists a row vector $\vec{c} \ge \mathbb{0}$ such that $\vec{c}A' = \vec{a}_0$ and $\vec{c}\,\vec{z}\,' \le z_0$. Let $z_1 = \vec{c}\,\vec{z}\,'$. Then $\vec{c}A\vec{x} \le \vec{c}\,\vec{z}\,'$ is the inequality $\vec{a}_0\vec{x} \le z_1$, a positive combination of the inequalities $A'\vec{x} \le \vec{z}\,'$. The inequality $\vec{a}_0\vec{x} \le z_0$ is this positive combination plus $\gamma(\mathbb{0}\vec{x} \le \vec{1})$, where γ is the nonnegative scalar $z_0 - z_1$. □

Here is another true statement for polyhedra, with conclusion closer to what we're looking for, but with an additional hypothesis:

Proposition 2. *An inequality that appears as a row of $A\vec{x} \le \vec{z}$ and that defines a nonempty face of $P(A,\vec{z})$ is redundant if and only if it can be written as a positive combination of other inequalities in the system.*

Proof. The "if" direction is still trivial. Suppose the inequality $\vec{a}_0\vec{x} \le z_0$ is redundant, and let $A'\vec{x} \le \vec{z}\,'$ be the system obtained by deleting that row. Then as above, there exists $\vec{c} \ge \mathbb{0}$ and $\gamma \ge 0$ such that the inequality $\vec{a}_0\vec{x} \le z_0$ is $\vec{c}A\vec{x} \le \vec{c}\,\vec{z}\,'$ plus $\gamma(\mathbb{0}\vec{x} \le \vec{1})$. But $\vec{c}A\vec{x} \le \vec{c}\,\vec{z}\,'$ is valid for P, and thus $\vec{c}\,\vec{z}\,'$ is greater than or equal to the maximum, over P, of the linear functional $\vec{c}A$. If $\gamma > 0$, then $\vec{c}\,\vec{z}\,' + \gamma$ is strictly greater than the maximum, over P, of the linear functional $\vec{c}A$. Therefore, no point in P satisfies $\vec{a}_0\vec{x} = z_0$, and thus the face defined by $\vec{a}_0\vec{x} \le z_0$ is empty, contradicting the hypothesis. We conclude that $\gamma = 0$. □

Finally, what the book was asking you to prove.

Proposition 3. *Suppose $P(A,\vec{z})$ is a d-polytope in $\mathbb{R}^d$. An inequality that appears as a row of $A\vec{x} \le \vec{z}$ is redundant if and only if it can be written as a positive combination of other inequalities in the system.*

Proposition 3 follows immediately from Proposition 1 and the following lemma.

Lemma 1. *If $P = P(A,\vec{z})$ is a d-polytope in $\mathbb{R}^d$, then $\mathbb{0}\vec{x} \le 1$ is a positive linear combination of inequalities in the system $A\vec{x} \le \vec{z}$.*

Proof. We may as well assume A has no zero rows. (If A has a zero row, then either the corresponding entry of $\vec{z}$ is zero, and we can delete that irrelevant row, or the corresponding entry of $\vec{z}$ is positive, in which case, we can scale that row to be $\mathbb{0}\vec{x} \le 1$ and we're done.)

If A is an $m \times d$ matrix, we are looking for a vector $\vec{c} \in (\mathbb{R}^m)^*$ with $\vec{c} \ge \mathbb{0}$ such that $\vec{c}A = \mathbb{0}$ and $\vec{c}\,\vec{z} = 1$. Equivalently, a vector $\vec{c}$ with $\vec{c} \ge \mathbb{0}$ such that $\vec{c}(A,\ \vec{z}) = (\mathbb{0},\ 1)$. Dualizing Farkas II, we see that this exists if and only if there does not exist a column vector $\vec{w}$ and a scalar s with $(A,\ \vec{z})\binom{\vec{w}}{s} \ge \vec{0}$ and $(\mathbb{0},\ 1)\binom{\vec{w}}{s} < 0$. Equivalently, there does not exist $\vec{w} \in \mathbb{R}^d$ and $t > 0$ such that $A\vec{w} - t\vec{z} \ge \vec{0}$, or in other words $A\vec{w} \ge t\vec{z}$. Equivalently (taking $\vec{v} = \vec{w}/t$) there does not exist $\vec{v} \in \mathbb{R}^d$ with $A\vec{v} \ge \vec{z}$.

If such a $\vec{v}$ does exist, let $\vec{x}$ be a vector in the interior of P, so that $A\vec{x} < \vec{z}$. (This is by Lemma 2.8 since A has no zero rows.) Since $A\vec{x} < \vec{z}$ and $A\vec{v} \ge \vec{z}$, we know that $\vec{x} \neq \vec{v}$. For any $\lambda \ge 1$, the vector $\vec{x}' = \lambda\vec{x} + (1-\lambda)\vec{v}$ has $A\vec{x}' \le \lambda\vec{z} + (1-\lambda)\vec{z} = \vec{z}$. Thus the vectors $\vec{x}'$ for $\lambda \ge 1$ constitute a ray in P, contradicting the hypothesis that P is a polytope. (Polytopes are bounded!) We conclude that the desired vector $\vec{c}$ exists. □

Part (ii). *I don't see how to do this with a Farkas lemma.* Suppose $P = P(A, \vec{z})$ is nonempty. As in the description of the problem, P is a d-polytope in $\mathbb{R}^d$. Suppose the inequality $\vec{a}\vec{x} \le z$ in the system is not redundant, and let $A'\vec{x} \le \vec{z}\,'$ be the system obtained by deleting the row $\vec{a}\vec{x} \le z$ from $A\vec{x} \le \vec{z}$. Then there is a point $\vec{x}$ in $P(A', \vec{z}\,')$ that is not in $P(A, \vec{z})$. We can take $\vec{x}$ to be in the interior of $P(A', \vec{z}\,')$. (If not, then $P(A, \vec{z})$ contains the interior of $P(A', \vec{z}\,')$. But then since $P(A, \vec{z})$ is closed, it contains $P(A', \vec{z}\,')$, which is a contradiction.)

Let $\vec{y}$ be a point in the interior of $P(A, \vec{z})$. Then the line segment $[\vec{x}, \vec{y}]$ intersects the hyperplane $H = \left\{\vec{x} \in \mathbb{R}^d : \vec{a}\vec{x} = z\right\}$ in a single point $\vec{y}\,'$. But then $\vec{y}\,'$ is in the interior of $P(A', \vec{z}\,')$, because both $\vec{x}$ and $\vec{y}$ are. (Easy exercise using Lemma 2.8: The interior of a polyhedron is convex. By the way, slightly harder but not hard: The interior of a convex set is convex.) Thus there is an open d-dimensional ball B containing $\vec{y}\,'$ and contained in $P(A', \vec{z}\,')$.

The face of $P(A, \vec{z})$ defined by $\vec{a}\vec{x} \le z$ is $\{\vec{x} \in P(A, \vec{z}) : \vec{a}\vec{x} = z\}$, which equals $P(A', \vec{z}\,') \cap H$, since the two systems differ only by the inequality $\vec{a}\vec{x} \le z$. But $P(A', \vec{z}\,') \cap H$ contains the $(d-1)$-dimensional set $B \cap H$, so the face is $(d-1)$-dimensional, or in other words, it is a facet.

Part (iii). The "only if" direction follows immediately from the definition of a face. Suppose the inequality $\vec{a}\vec{x} \le z$ is valid for P and has $\vec{a}\vec{x}_F = z$. Since it is valid for P, it is valid for F. By Lemma 2.9, we see that $\vec{a}\vec{x} = z$ for all $\vec{x} \in F$. (Recall that in class we explained why the conditions of Lemma 2.9 are also equivalent to the assumption that $\vec{y}$ is in the relative interior of P. We apply the Lemma with $P = F$ and $\vec{y} = \vec{x}_F$, and conclude that condition (ii) of the lemma holds.) Since F is $(d-1)$-dimensional $H = \left\{\vec{x} \in \mathbb{R}^d : \vec{a}\vec{x} = z\right\}$ is a hyperplane containing F, we conclude that $\operatorname{aff}(F) = H$, so that $F = P \cap H$ by Proposition 2.3(iv). In other words, the inequality $\vec{a}\vec{x} \le z$ defines F.

Part (iv). Certainly, any nontrivial inequality can be scaled in that way. Suppose there are two such linear inequalities, $\vec{a}\vec{x} \le z_1$ and $\vec{b}\vec{x} \le z_2$. As in (iii), both are satisfied with equality on the same hyperplane $\operatorname{aff}(F)$. Thus they are multiples of each other. They also point the same direction so one is a positive mulitple of the other. But if that multiple is γ, then

$$1 = \sum_{i=1}^{d} |b_i| = \sum_{i=1}^{d} |\gamma a_i| = \gamma \sum_{i=1}^{d} |a_i| = \gamma.$$

The rest of the problem. Suppose there is an irredundant description of a d-polytope $P \subset \mathbb{R}^d$ as $P(A, \vec{z})$. By (ii), each of the inequalities defines a facet. By (iv), each defines a distinct facet.

Suppose some facet F is not defined by an inequality in the system. Choose a point $\vec{x}_F$ in the relative interior of F. By (iii), none of the defining inequalities is satisfied with equality at $\vec{x}_F$. Thus $\vec{x}_F$ is some positive distance from the hyperplane for each defining inequality. Since there are only finitely many inequalities in the system, there is an open ball B about $\vec{x}_F$ that is disjoint from all of these hyperplanes, and thus since $\vec{x}_F \in P$, the entire ball B is in P. Now let $\vec{a}\vec{x} \le z$ be a valid inequality that defines F. Although it is not in the system, by definition of a face, such an inequality exists. But then by (iii), $\vec{a}\vec{x}_F = z$, so the inequality $\vec{a}\vec{x} \le z$ cannot hold on the entire d-dimensional ball $B \subset P$, contradicting the assumption that $\vec{a}\vec{x} \le z$ is valid.

We have showed that an irredundant system of linear inequalities defining a d-polytope in $\mathbb{R}^d$ contains exactly one facet-defining inequality for each facet, and contains no other inequalities. The statement also follows from Proposition 2.2(i) because polarization takes vertices to facet-defining inequalities.

If $\dim(P) < d$, then facet-defining inequalities are no longer unique up to scaling. Instead, there is are $d - \dim(P)$ additional degrees of freedom. The same level of uniqueness can be recovered by requiring that the $\vec{a}$ in a facet-defining hyperplane $\vec{a}\vec{x} \le z$ lie in the subspace of $(\mathbb{R}^d)^*$ that is dual to $\operatorname{aff}(P)$.