MA 724
Homework 4, Comments and some solutions.

Lecture 2, Problem 0

Let P be a polyhedron and let F_0 be minimal among nonempty faces of P. We are to show that for any $\vec{x}_0 \in F_0$, the lineality space of P is $\mathrm{lin}(P) = F_0 - \vec{x}_0$. Here $F_0 - \vec{x}_0$ means $\{\vec{x} - \vec{x}_0 : \vec{x} \in F_0\}$. In particular, $F_0 - \vec{x}_0$ is a linear subspace. The "Thus,..." statement in the book follows immediately.

One nice proof consists of using Additional Problems 1 and 2 from the previous assignment and then using a Farkas Lemma.

Translating the polyhedron does not change the lineality space, so we may as well translate to make $\vec{x}_0 = \vec{0}$. By Additional Problems 1 and 2, we need to show that $F_0 = \left\{\vec{y} \in \mathbb{R}^d : t\vec{y} \in F_0 \text{ for all } t \in \mathbb{R}\right\}$. On one hand, $\left\{\vec{y} \in \mathbb{R}^d : t\vec{y} \in F_0 \text{ for all } t \in \mathbb{R}\right\} \subseteq \left\{\vec{y} \in \mathbb{R}^d : 1\vec{y} \in F_0\right\} = F_0$.

Suppose there exists $\vec{y} \in F_0$ such that $\vec{y} \notin \mathrm{lin}(F_0)$. That means that there exists $t \in \mathbb{R}$ such that $t\vec{y} \notin F_0$. In particular, $\vec{y} \neq \vec{0}$. By Farkas IV, there exists a linear inequality $\vec{a}\vec{x} \leq \alpha$ that is valid for F_0 but not satisfied by $t\vec{y}$. (This interpretation of Farkas IV is found in the paragraph before Proposition 1.10 and was also given in class.) Let $\alpha' = \max_{\vec{x} \in F_0} \vec{a}\vec{x}$. This maximum is attained at a unique point on the line segment $[\vec{0}, t\vec{y}]$ since $\vec{0} \in F_0$, since $\vec{a}\vec{x} \leq \alpha$ is valid for F_0 and since $\vec{a}(t\vec{y}) > \alpha$. The face $G = \{\vec{x} \in F_0 : \vec{a}\vec{x} = \alpha'\}$ is a proper, nonempty face of F_0, because it contains only one point on $[\vec{0}, t\vec{y}]$, but F_0 contains the entire line segment $[\vec{0}, y] \subset [\vec{0}, t\vec{y}]$. This contradiction shows that $F_0 = \left\{\vec{y} \in \mathbb{R}^d : t\vec{y} \in F_0 \text{ for all } t \in \mathbb{R}\right\}$.

If a polyhedron P has a vertex, then that vertex has lineality space $\{\vec{0}\}$, so by Additional Problem 2 from last time, P has lineality space $\{\vec{0}\}$.

Lecture 2, Problem 2

Let $P = P(A, \vec{z})$ be a polytope, for $A \in \mathbb{R}^{m \times d}$ and $\vec{z} \in \mathbb{R}^m$. Then the map $\eta : \vec{x} \mapsto (A\vec{x} - \vec{z})$ is an affine map from $\mathbb{R}^d$ to $\mathbb{R}^m$, so its image is an affine subspace. Now, $\vec{x} \in \mathbb{R}^d$ is in P if and only if $\eta(\vec{x}) \leq \vec{0}$, so $\eta(P)$ is intersection of the affine subspace $\eta(\mathbb{R}^d)$ with the orthant $\left\{\vec{y} \in \mathbb{R}^m : \vec{y} \leq \vec{0}\right\}$. If $\ker(A)$ has a nonzero vector $\vec{w}$, then the line spanned by $\vec{w}$ is in the lineality space of P (check the defining inequalities!), contradicting the assumption that P is a polytope (and thus bounded). So η is one-to-one, and therefore it restricts to a bijection between P and $\eta(P)$. Thus P is affinely equivalent to $\eta(P)$. Since P is bounded, and $\eta(P)$ is affinely equivalent to P, $\eta(P)$ is also bounded.

Lecture 2, Problem 4

You should have used Theorem 2.7, but of course not any part of that theorem that was proved using duality (the "coatomic" property and the fact that $L(P)^{\mathrm{op}}$ is the face lattice of a polytope). Luckily, the rest of the thoerem is enough:

Let F be a proper (!) face of P and consider a maximal chain in $[F, P] \subseteq L(P)$. Since $L(P)$ is graded and the rank function is dimension + 1, the element covered by P in the chain has dimension $\dim(P) - 1$, or in other words it is a facet.

Lecture 2, Problem 7

Interpret each column of $M(P)$ as the set of vertices of a facet of P. Then take all intersections of these sets (i.e. for each set of columns, take the intersection). Since $L(P)$ is coatomic, every face's vertex set appears in this way. Then order these vertex sets by containment to get $L(P)$. The dimension is obtained as the length of the longest chain minus one. The matrix $M(P^{\triangle})$ is the transpose of $M(P)$.

Comment: All of you should have checked your solution against a small example. This is **always** a good idea. You might have assumed that, in a d-polytope, the intersection of two facets is a face of dimension $d - 2$. This is not necessarily true. In this case, a square would have been an instructive example.

Another comment: Another common mistake is trying to build the face lattice "from the bottom up." That's a lot harder. Why? Because we know the meet operation is intersection, and that interacts well with considering vertex sets of faces. Is there a similarly simple expression for the join operation? For example, what is the join of two vertices of a 3-cube?