MA 724
Homework 3, Comments and some solutions.

Chapter 1, Problem 3

I think Ziegler was looking for a constructive solution to this problem, i.e. given $C = \text{cone}(W)$, determine V and Y so that the set $\left\{\vec{x} \in \mathbb{R}^d : \begin{pmatrix}1\\ \vec{x}\end{pmatrix} \in C\right\}$ equals $\text{conv}(V) + \text{cone}(Y)$. I don't see how to do that, but I admit I have not put hours and hours into it. But we can do this problem just by appealing to the results in Lecture 1, and that's the point I was trying to get across when I assigned this. (But, good job to one of you for finding a nice way to do this!) Here goes:

By the Main Theorem, C is also an H-polyhedron, so $\left\{\begin{pmatrix}1\\ \vec{x}\end{pmatrix} \in \mathbb{R}^{d+1} : \begin{pmatrix}1\\ \vec{x}\end{pmatrix} \in C\right\}$ is also an H-polyhedron. (Take the defining inequalities for C and adjoin the inequalities $x_0 \leq 1$ and $-x_0 \leq 1$.) Now the main Theorem says that $\left\{\begin{pmatrix}1\\ \vec{x}\end{pmatrix} \in \mathbb{R}^{d+1} : \begin{pmatrix}1\\ \vec{x}\end{pmatrix} \in C\right\}$ is a V-polyhedron $\text{conv}(V) + \text{cone}(Y)$. Each vector in V and Y has 1 as it's 0th coordinate, so we can write $\left\{\vec{x} \in \mathbb{R}^d : \begin{pmatrix}1\\ \vec{x}\end{pmatrix} \in C\right\}$ as a V-polyhedron in $\mathbb{R}^d$ by deleting all the 0th coordinates from the vectors in V and the vectors in Y.

Chapter 1, Problem 4

Proposition 1. *Given an $m \times d$ matrix A and $\vec{z} \in \mathbb{R}^m$, then*
***Either** there exists a vector $x \in \mathbb{R}^d$ with $A\vec{x} \leq \vec{z}$ and $\vec{x} \geq \vec{0}$,*
***or** there exists $\vec{c} \in (\mathbb{R}^n)^*$ with $\vec{c} \geq \mathbb{0}$, $\vec{c}A \geq \mathbb{0}$, and $\vec{c}\vec{z} < 0$,*
***but not both**.*

Proof.

$$\exists\, \vec{x} \in \mathbb{R}^d \text{ with } A\vec{x} \leq \vec{z},\ -\vec{x} \leq \vec{0} \iff \exists\, \vec{x} \in \mathbb{R}^d \text{ with } \begin{pmatrix}-I\\ A\end{pmatrix}\vec{x} \leq \begin{pmatrix}\vec{0}\\ \vec{z}\end{pmatrix}$$

$$\text{(by Farkas I)} \iff \nexists\, (\vec{b},\vec{c}) \in (\mathbb{R}^{d+n})^* \text{ with } (\vec{b},\vec{c}) \geq \mathbb{0},\ (\vec{b},\vec{c})\begin{pmatrix}-I\\ A\end{pmatrix} = \mathbb{0},\ (\vec{b},\vec{c})\begin{pmatrix}\mathbb{0}\\ \vec{z}\end{pmatrix} < 0$$

$$\iff \nexists\, (\vec{b},\vec{c}) \in (\mathbb{R}^{d+n})^* \text{ with } (\vec{b},\vec{c}) \geq \mathbb{0},\ -\vec{b} + \vec{c}A = \mathbb{0},\ \vec{c}\vec{z} < 0$$

$$\iff \nexists\, \vec{c} \in (\mathbb{R}^n)^* \text{ with } \vec{c} \geq \mathbb{0},\ \vec{c}A \geq \mathbb{0},\ \vec{c}\vec{z} < 0$$

□

This makes sense: either there is a solution with nonnegative coefficients or there is a positive linear combination of the defining inequalities which is obviously false for vectors with nonnegative coefficients.

Chapter 3, Problem 5

Proof. Suppose $A \in \mathbb{R}^{m\times d}$, $B \in \mathbb{R}^{m\times d}$, and $C \in \mathbb{R}^{p\times d}$, so that $\vec{u} \in \mathbb{R}^m$, $\vec{v} \in \mathbb{R}^n$, and $\vec{w} \in \mathbb{R}^p$.

$$\exists\, \vec{x} \in \mathbb{R}^d \text{ with } A\vec{x} = \vec{u},\ B\vec{x} \geq \vec{v},\ C\vec{x} \leq \vec{w} \iff \exists\, \vec{x} \in \mathbb{R}^d \text{ with } \begin{pmatrix}A\\ -A\\ -B\\ C\end{pmatrix}\vec{x} \leq \begin{pmatrix}\vec{u}\\ -\vec{u}\\ -\vec{v}\\ \vec{w}\end{pmatrix}$$

$$\text{(by Farkas I)} \iff \nexists\, (\vec{a}_1,\vec{a}_2,-\vec{b},\vec{c}) \in (\mathbb{R}^{2m+n+p})^* \text{ with } (\vec{a}_1,\vec{a}_2,-\vec{b},\vec{c}) \geq \mathbb{0},$$

$$(\vec{a}_1,\vec{a}_2,-\vec{b},\vec{c})\begin{pmatrix}A\\ -A\\ -B\\ C\end{pmatrix} = \mathbb{0},\ (\vec{a}_1,\vec{a}_2.-\vec{b},\vec{c})\begin{pmatrix}\vec{u}\\ -\vec{u}\\ -\vec{v}\\ \vec{w}\end{pmatrix} < 0$$

$$\text{(setting } \vec{a} = \vec{a}_1 - \vec{a}_2) \iff \nexists\, (\vec{a},\vec{b},\vec{c}) \in (\mathbb{R}^{m+n+p})^* \text{ with } \vec{b} \leq \mathbb{0},\ \vec{c} \geq \mathbb{0},$$

$$\vec{a}A + \vec{b}B + \vec{c}C = \mathbb{0},\ \vec{a}\vec{u} + \vec{b}\vec{v} + \vec{c}\vec{w} < 0$$

□

Additional problem 1

There were several ways to argue this. Here are three. In every case, the "easy" direction is exactly the same.

Sample solution 1. By definition, $\mathrm{lin}(P)$ equals $\left\{\vec{y} \in \mathbb{R}^d : \vec{x} + t\vec{y} \in P \text{ for all } \vec{x} \in P,\, t \in \mathbb{R}\right\}$. Fix $x_0 \in P$ and let U be the set $\left\{\vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in P \text{ for all } t \in \mathbb{R}\right\}$. The inclusion $\mathrm{lin}(P) \subseteq U$ is immediate. Suppose $\vec{y} \in U$. Given any point $\vec{x} \in P$ and any $t \in \mathbb{R}$, let L be the line segment $[\vec{x}_0, \vec{x} + t\vec{y}]$. All points on L are of the form $(1-\lambda)\vec{x}_0 + \lambda(\vec{x} + t\vec{y})$ for $0 \le \lambda \le 1$. For each λ with $0 \le \lambda < 1$, the point $\vec{z} = (1-\lambda)\vec{x}_0 + \lambda(\vec{x} + t\vec{y})$ equals $(1-\lambda)(\vec{x}_0 + t'\vec{y}) + \lambda\vec{x}$, where $t' = t\frac{\lambda}{1-\lambda}$. Since P is convex and since $\vec{x}$ and $\vec{x}_0 + t'\vec{y}$ are both in P, we see that $\vec{z} \in P$. Thus all of L, except possibly $\vec{x} + t\vec{y}$ is in P. But P is closed, so $\vec{x} + t\vec{y} \in P$ as well. We see that $\vec{y} \in \mathrm{lin}(P)$. □

Sample solution 2. By definition, $\mathrm{lin}(P)$ equals $\left\{\vec{y} \in \mathbb{R}^d : \vec{x} + t\vec{y} \in P \text{ for all } \vec{x} \in P,\, t \in \mathbb{R}\right\}$. Fix $x_0 \in P$ and let U be the set $\left\{\vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in P \text{ for all } t \in \mathbb{R}\right\}$. The inclusion $\mathrm{lin}(P) \subseteq U$ is immediate.

On the other hand, suppose $\vec{y} \notin \mathrm{lin}(P)$. Then there exists $\vec{x} \in P$ and $t \in \mathbb{R}$ such that $\vec{x} + t\vec{y} \notin P$. Without loss of generality (perhaps by replacing $\vec{y}$ with $-\vec{y}$, which we can do because if $\vec{y} \in U$ then $-\vec{y} \in U$), we can assume that $t > 0$. Writing $P = P(A, \vec{z})$, there is a row $\vec{a}_i\vec{x} \le z_i$ of the inequalities $A\vec{x} \le \vec{z}$ with $\vec{a}_i(\vec{x} + t\vec{y}) > z_i$. Since $\vec{x} \in P$, we have $\vec{a}_i\vec{x} \le z_i$, and we conclude that $\vec{a}_i(t\vec{y}) > 0$, and since $t > 0$, we see that $\vec{a}_i\vec{y} > 0$. In particular, for large enough $t' > 0$, $\vec{a}_i(\vec{x}_0 + t'\vec{y}) > z_i$, so $\vec{y} \notin U$. □

Here is a nice way to explain it that I learned from a student.

Sample solution 3. By definition, $\mathrm{lin}(P)$ equals $\left\{\vec{y} \in \mathbb{R}^d : \vec{x} + t\vec{y} \in P \text{ for all } \vec{x} \in P,\, t \in \mathbb{R}\right\}$. Fix $x_0 \in P$ and let U be the set $\left\{\vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in P \text{ for all } t \in \mathbb{R}\right\}$. The inclusion $\mathrm{lin}(P) \subseteq U$ is immediate.

Conversely, suppose $\vec{y} \in U$. That is, $\vec{x}_0 + t\vec{y} \in P$ for all $t \in \mathbb{R}$. Writing $P = P(A, \vec{z})$, notice that $A\vec{y} = \vec{0}$: If $A\vec{y}$ has some positive entry (say in position i), then we can choose t large enough to make $\vec{a}_i(x_0 + t\vec{y}) \le z_i$ false, and if $A\vec{y}$ has a negative entry in position i then we can choose t negative with large enough absolute value to make $\vec{a}_i(x_0 + t\vec{y}) \le z_i$ false. Thus for all $\vec{x} \in P$, we have $A(\vec{x} + t\vec{y}) = A\vec{x} \le \vec{z}$, so $(\vec{x} + t\vec{y}) \in P$. □

Additional Problem 2

Let P be a polyhedron and let F be a nonempty face of P. We are to show that $\mathrm{lin}(F) = \mathrm{lin}(P)$. Choosing $x_0 \in F$ and using the identities $\mathrm{lin}(P) = \left\{\vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in P \text{ for all } t \in \mathbb{R}\right\}$ and $\mathrm{lin}(F) = \left\{\vec{y} \in \mathbb{R}^d : \vec{x}_0 + t\vec{y} \in F \text{ for all } t \in \mathbb{R}\right\}$ from Additional Problem 1, we see that $\mathrm{lin}(F) \subseteq \mathrm{lin}(P)$. On the other hand, suppose $\vec{y} \in \mathrm{lin}(P)$. We already know $\vec{0} \in \mathrm{lin}(F)$, so assume $\vec{y} \ne \vec{0}$. Let H be a hyperplane defining F as a face of P. Since H is associated to a valid inequality for P, all of P is contained on one side of H. The line $\{\vec{x}_0 + t\vec{y} : t \in \mathbb{R}\}$ must be contained in the hyperplane H. Otherwise, the line contains points on both sides of H, so the line contains points not in P. That would contradict the supposition that $\vec{y} \in \mathrm{lin}(P)$. Thus $\vec{x}_0 + t\vec{y}$ is in $P \cap H$ for all t. But $P \cap H = F$, and we have shown that $\vec{y} \in \mathrm{lin}(F)$.