MA 724
Homework 2, Comments and some solutions.

Chapter 0, Problem 3

I graded this out of 4 points each for the 4 different things you had to prove.

Dimension. Since Π_{d-1} is contained in the hyperplane $\left\{\vec{x} \in \mathbb{R}^d : \sum_{i=1}^d = \binom{d+1}{2}\right\}$, its dimension is at most $d-1$. We will show that there are d linearly independent vertices of Π_{d-1} These will then be affinely independent, so that their affine span has at least dimension $d-1$. This is trivial for $d = 1$, so assume $d > 1$. Take the vertices in the permutohedron whose last coordinate is d. These are obtained from the vertices of Π_{d-2} by adjoining a new last coordinate d. By induction, there are $d-1$ linearly independent vertices of Π_{d-2}, and when we adjoin a new last coordinate d, we obtain $d-1$ linearly independent vectors in $\mathbb{R}^d$, all of which are vertices of Π_{d-1}. Each of these new vectors has $x_1 + \cdots + x_{d-1} = \binom{d}{2}$ and $x_d = d$, so they are contained in the linear subspace of points satisfying $x_1 + \cdots + x_{d-1} - \frac{d-1}{2}x_d = 0$. It's now easy to find a vertex of Π_{d-1} not contained in this subspace, for example, the point $\vec{x}$ with $x_1 = n$, $x_2 = n-1$, etc., which has $x_1 + \cdots + x_{d-1} - \frac{d-1}{2}x_d = \left[\binom{d+1}{2} - 1\right] - \frac{d-1}{2} = \frac{d^2-1}{2}$, which is nonzero for $d > 1$.

Comment: Many of you did this by showing that the vertex for the identity permutation, and all its $d-1$ neighbors, constitute a linearly (and thus affinely) independent set. That's a fine way to do it, but notice that it's much easier to subtract the coordinates of the identity permutation from all of them, and show that the $d-1$ neighbors are linearly independent. Also, notice that if you want to show that all d vectors form an independent set, it's much easier to use *column* operations in this case.

Facets. We will show that the facets are in bijection with the nonempty proper subsets of $[d]$. Given such a subset, the corresponding inequality is $\sum_{i\in S} x_i \geq \binom{|S|+1}{2}$. All vertices of Π_{d-1} (and thus all points of Π_{d-1} satisfy this inequality, because the smallest that $\sum_{i\in S}$ can be is $1 + 2 + \cdots + |S| = \binom{|S|+1}{2}$. This is attained for all vertices having the values 1 through $|S|$ in their coordinates indexed by indices in S. The convex hull of these vertices is a $(d-2)$-polytope isomorphic to $\Pi_{|S|-1} \times \Pi_{d-|S|-1}$, so these inequalities define facets. It only remains to show that there are no other facets.

The facet defined by S contains the vertex $\vec{x}_0 = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ d \end{pmatrix}$ if and only if $S = \{1, \ldots, i\}$ for some i with $1 \leq i < n$. If $S = \{1, \ldots, i\}$ and $S' = \{1, \ldots j\}$ with $i < j$, then the intersection of the facets defined by S and S' contains all vertices with the values 1 through i appearing as the first through ith coordinates, and $i+1$ through j appearing as the $(i+1)$st through jth coordinates. Arguing similarly, we see that the intersection of all of these facets containing $\vec{x}_0$, except the facet defined by $\{1, \ldots, i\}$, has exactly two vertices: $\vec{x}_0$ and the point $\vec{x}_i$ whose ith coordinate is $i+1$, whose $(i+1)$st coordinate is i, and whose jth coordinate is j for $j \notin \{i, i+1\}$ (as asserted in Ziegler). Consider the polyhedron P defined only by the facet-defining inequalities for facets, defined by subsets, containing $\vec{x}_0$. Then P contains Π_{d-1}, and P is $\vec{x}_0 + \operatorname{cone}(\{\vec{x}_i - \vec{x}_0 : i \in [d]\})$. If there are any additional facets containing $\vec{x}_0$ besides those defined by subsets, then these facets define a polyhedron P' strictly smaller than P. This P' is $\vec{x}_0 + \operatorname{cone}(Y)$ for some set of vectors that cannot contain $\{\vec{x}_i - \vec{x}_0 : i \in [d]\}$, since P' is strictly smaller than P. In particular, some $\vec{x}_i$ is not in Π_{d-1} and this contradiction shows that the only facets containing $\vec{x}_0$ are those defined by subsets. By the symmetry of permuting coordinates, we conclude that all facets of P are defined by subsets as above.

Simple. Since we know what the facets are, we notice that the facet defined by S contains the vertex labeled by the identity permutation if and only if $S = \{1, \ldots, i\}$ for some i with $1 \leq i < n$. There are exactly $d-1$ facets containing that vertex, and all vertices are symmetric by the symmetry of permuting coordinates, so Π_{d-1} is simple.

Zonotope. One way to do this is on p. 200 of Ziegler. We'll show another, less slick way.

We'll project the cube $[0,1]^{\binom{d}{2}} \subset \mathbb{R}^{\binom{d}{2}}$ to $\mathbb{R}^d$ and show that the image is Π_{d-1}. Label the standard unit basis vectors for $\mathbb{R}^{\binom{d}{2}}$ as $\vec{e}_{ij}$ for $1 \leq i < j \leq d$ and define a linear map $\varphi : \mathbb{R}^{\binom{d}{2}} \to \mathbb{R}^d$ by sending e_{ij} to $e_i - e_j$. Define an affine map $\bar{\varphi}$ sending $\vec{x}$ to $\varphi(x) + \vec{x}_0$.

To show that the image of the cube under $\bar{\varphi}$ is Π_{d-1}, we need to show two things: First, we'll show that every vertex of Π_{d-1} is the image of some vertex of the cube under $\bar{\varphi}$. That implies that Π_{d-1} is contained in the image. Then we'll show that each vertex of the cube projects into Π_{d-1}, by showing that the projections satisfy all of the facet-defining inequalities. That implies that the image is contained in Π_{d-1}.

Proposition 1. *If $\vec{x}$ is a vertex of Π_{d-1}, then there exists a vertex $\vec{v}$ of $[0,1]^{\binom{d}{2}}$ such that $\bar{\varphi}(\vec{v}) = \vec{x}$. Specifically, $\vec{v}$ is the vector in $\{0,1\}^{\binom{d}{2}}$ whose ij-coordinate is 1 if and only if $x_i > x_j$.*

Proof. We argue by induction on the number of inversions of the permutation whose one-line notation is $x_1 \cdots x_n$. If there are no inversions, then $\vec{x} = \vec{x}_0 = \bar{\varphi}(\vec{0})$. Otherwise, there exists some $k \in [d-1]$ such that k precedes $k+1$ in $x_1 \cdots x_n$. Let $\vec{x}'$ be the vector obtained by replacing k by $k+1$ and $k+1$ by k in the coordinates of $\vec{x}$. By induction, the vector $\vec{v}'$ in $\{0,1\}^{\binom{d}{2}}$ whose ij-coordinate is 1 if and only if $x'_i > x'_j$ has $\bar{\varphi}(\vec{v}') = \vec{x}'$. Now $\bar{\varphi}(\vec{v}) = \bar{\varphi}(\vec{v}') + e_i - e_j = \vec{x}$, where i is the position of $k+1$ in $\vec{x}$ and j is the position of k in $\vec{x}$. □

Proposition 2. *If $\vec{v}$ is a vertex of $[0,1]^{\binom{d}{2}}$, then $\bar{\varphi}(\vec{v}) \in \Pi_{d-1}$.*

Proof. First, we show that $\bar{\varphi}(\vec{v})$ satisfies the facet-defining inequalities for facets containing $\vec{x}_0$, as described above. This amount to showing that, for each $i \in [d-1]$, the sum of the first i coordinates of $\bar{\varphi}(\vec{v})$ is at least $\binom{i+1}{2}$. We argue by induction on the number of 1's among the coordinates of $\vec{v}$. If there are no 1's, then $\bar{\varphi}(\vec{v}) = \vec{x}_0$, which satisfies the inequalities (with equality, in fact). Otherwise, suppose the ij-coordinate of $\vec{v}$ is 1 for some $1 \le i < j \le d$, and let $\vec{v}'$ be $\vec{v} - \vec{e}_{ij}$, the vector obtained by changing that 1 to 0. Then $\bar{\varphi}(\vec{v}') = \bar{\varphi}(\vec{v}) - \vec{e}_i + \vec{e}_j$, so for any $k \in [d-1]$, the sum of the first k coordinates of $\bar{\varphi}(\vec{v})$ is at least as big as the sum of the first k coordinates of $\bar{\varphi}(\vec{v}')$. By induction, the inequalities hold for $\bar{\varphi}(\vec{v}')$, so the inequality holds for $\bar{\varphi}(\vec{v})$ as well.

To see that the other inequalities hold, make the same inductive argument with the following modifications: Instead of $\vec{x}_0$, take any vertex $\vec{x}$ of Π_{d-1}. Let $\vec{v}$ be the vector in $[0,1]^{\binom{d}{2}}$ such that $\bar{\varphi}(\vec{v}) = \vec{x}$, which exists by the previous proposition. Let $\vec{w}$ be any other vector in $[0,1]^{\binom{d}{2}}$. Instead of doing induction on the number of zeros in $\vec{w}$, do induction on the number of positions where $\vec{w}$ differs from $\vec{v}$. When $\vec{v}$ and $\vec{w}$ differ at some position ij, either subtract or add $\vec{e}_{ij}$ to make $\vec{w}$ closer to $\vec{v}$. Instead of considering the first k positions, consider the positions where the coordinates of $\vec{v}$ are 1 through k. □

Chapter 0, Problem 9

This was really 2 separate parts, so I graded it out of 8 (4 each).

The "show bijectively" question could be straight out of MA 524, although I think that without Gale's evenness criterion for context, it would have been harder to understand what was being asked. There are many nice ways to do this.

Proposition 3. *The number of subsets $S \subseteq [n]$ such that S is a disjoint union of k pairs of the form $\{i, i+1\}$ is $\binom{n-k}{k}$.*

Proof. Here is a bijection η from such subsets S to sequences of dots and rectangles with exactly k rectangles and $n-2k$ dots: Write the numbers $1, \ldots, n$ on a horizontal line. For each pair $\{i, i+1\}$ in S, draw a rectangle around i and $i+1$ and erase the numbers i and $i+1$. Then replace each remaining number with a dot. The resulting sequence is $\eta(S)$. (For example, for $n = 9$ and $S = \{2,3,4,5,7,8\}$, we would write $\bullet$ ▭ ▭ $\bullet$ ▭ $\bullet$.

The inverse map θ is as follows. Given a sequence T of dots and rectangles, starting from the left, write the numbers $1, \ldots, n$, one number replacing each dot and two numbers inside each rectangle. Then $\theta(T)$ is the set of numbers occurring inside rectangles.

Sequences of dots and rectangles with exactly k rectangles and $n-2k$ dots are counted by $\binom{n-k}{k}$ because for each of the $n-k$ positions in the sequence, we choose exactly k of them to be filled with rectangles. □

Now, we can count facets of the cyclic polytope $C_d(n)$. These are subsets that satisfy Gale's evenness criterion.

Proposition 4. *The number of facets of $C_d(n)$ is $\frac{n}{n-k}\binom{n-k}{k}$ if $d = 2k$ or $2\binom{n-k-1}{k}$ if $d = 2k+1$.*

Proof. The vertices of $C_d(n)$ are $\{\vec{x}(t_1), \ldots, \vec{x}(t_n)\}$ with $t_1 < \cdots < t_n$. The facets of $C_d(n)$ are the convex hulls of sets $\{\vec{x}(t_i) : i \in S\}$ for sets S satisfying Gale's evenness criterion. (We will call such sets "Gale-even sets" in this proof.) A Gale-even set S is not necessarily a disjoint union of pairs of the form $\{i, i+1\}$. Instead it is a disjoint union of sets $\{i, i+1\}$ and possibly a set $\{1\}$ and/or a set $\{n\}$.

If $d = 2k$, then Gale-even sets come in two kinds: Either S is a disjoint union of k subsets $\{i, i+1\}$ of $[n]$ or S is a disjoint union of $\{1, n\}$ and $k-1$ subsets $\{i, i+1\}$ of $\{2, \ldots, n-1\}$. By Proposition 3, the first kind of subset is counted by $\binom{n-k}{k}$ and the second kind of subset is counted by $\binom{n-2-(k-1)}{k-1} = \binom{n-1-k}{k-1}$. The total number of Gale-even sets is

$$\binom{n-k}{k} + \binom{n-1-k}{k-1} = \binom{n-k}{k} + \frac{(n-k-1)!}{(k-1)!(n-2k)!} = \binom{n-k}{k} + \frac{k}{n-k} \cdot \frac{(n-k)!}{k!(n-2k)!} = \frac{n}{n-k}\binom{n-k}{k}.$$

If $d = 2k-1$, then Gale-even sets again come in two kinds. This time, either S is a disjoint union of the set $\{1\}$ with k subsets $\{i, i+1\}$ of $\{2, \ldots, n\}$ or S is a disjoint union of $\{n\}$ with k subsets $\{i, i+1\}$ of $\{1, \ldots, n-1\}$. Proposition 3 implies that each kind of set is counted by $\binom{n-1-k}{k}$, for a total of $2\binom{n-k-1}{k}$. □

It is straightforward to go from Proposition 4 to the formula in the book using floors and ceilings, but I don't think that is an improvement. What is nice about the formula with floors and ceilings is that it is a huge hint for how you prove Proposition 4, by breaking up the count as a sum of two particular binomial coefficients and referring to Proposition 3.

For the estimates, I guess we could think hard about approximating binomial coefficients, but I just put them into Maple: $C_{10}(20) = 4004 \approx 4 \cdot 10^3$, $C_{10}(100) = 60990020 \approx 6 \cdot 10^7$, and $C_{50}(100) = 70118062854865191504 \approx 7 \cdot 10^{19}$.