MA/CSC 416, Midterm exam, Comments and some solutions.

Short Answer Section

I am adding a few explanations, although you were not required to explain.

Question. How many nonnegative integer solutions are there to the inequality below? (Notice that there are two "$\leq$" signs.)

$$4 \leq x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 6$$

Answer: $\binom{10}{4} + \binom{11}{5} + \binom{12}{6}$.
The set of solutions breaks up into three pieces: solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 4$, solutions to $x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 5$, and solutions to $x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 6$. People had more trouble with this than I expected, and I realize that this was calling on you to remember a formula, possibly without good reasons to remember it. So, when I graded this, I gave at least 4 points for showing an attempt to use the Second Counting Principle to count the $= 4$, $= 5$, and $= 6$ cases and then add.

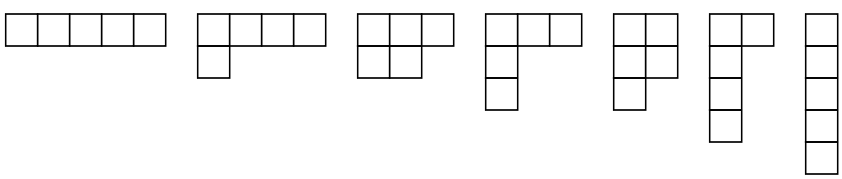
Another answer: $\binom{13}{7} - \binom{10}{7}$. This counts solutions of $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 6$ and subtracts off the number of those solutions that satisfy $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 3$.

Question. Consider the number 510510 with prime factorization $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$. What is the sum of the positive-integer divisors of 510510?

Answer: $3 \cdot 4 \cdot 6 \cdot 8 \cdot 12 \cdot 14 \cdot 18$.
This is a special case of what you proved in Section 1.7, Problem 19.

Question. Draw the Ferrers diagrams of all partitions of 5.

Answer:



Question. List all set partitions of [5] with exactly 4 blocks.
The point here was to choose 2 elements to be together in one block. The other elements are in singleton blocks.

Answer:

12|3|4|5	13|2|4|5	14|2|3|5	15|2|3|4
1|23|4|5	1|24|3|5	1|25|3|4	
1|2|34|5	1|2|35|4		
1|2|3|45			

Question. Write down all of the one-to-one functions in $F_{3,3}$ as ordered 3-tuples (i.e. ordered triples). Then circle all of the functions that are also onto.

Answer: Here are all the one-to-one functions, and you should have circled *all* of them, because a function from a set to itself is one-to-one if and only if it is onto. (Or, in this small example, you could just verify that they are all onto.)

$$\begin{array}{cc} (1,2,3) & (1,3,2) \\ (2,1,3) & (2,3,1) \\ (3,1,2) & (3,2,1) \end{array}$$

Problems requiring short explanations

For each problem, I am giving a sample answer showing the kind of explanation that was necessary. Then I add some comments in italics. The italicized comments are not part of the suggested answer.

Question. Let r and n be positive integers. Write an expression for the number of pairs (A, B) such that A is a subset of $[n]$ with $|A| = r$, and B is a subset of A with no restriction on the size of B.

Answer: Use the Fundamental Counting Principle. There are $\binom{n}{r}$ ways to choose A and 2^r ways to choose B. So the expression is $\binom{n}{r}2^r$. *This was similar to the left sides of the identities in Homework 2, Section 1.2, Problem 10.*

Question. How many 15-letter words are there that have a total of 4 A's, 5 B's, 3 C's and 3 D's **and** end in A.

Answer: Since the last letter has to be A, we can make one of these words by first making a word with 3 A's, 5 B's, 3 C's and 3 D's, then putting an A at the end. There are $\binom{14}{3,5,3,3}$ ways to do that. *This recalls the reasoning involved in Homework 2, Section 1.2, Problem 9b.*

Question. Prove that $5^n = \sum_{i=0}^{n} \binom{n}{i} 4^i$.

Answer: The Binomial Theorem says $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$. Substitute $x = 4$ and $y = 1$ and you get $5^n = \sum_{i=0}^{n} \binom{n}{i} 4^i$. *This builds on the idea learned in Homework 4, Section 1.7, Problem 6 and stressed in class: The idea of specializing a polynomial identity to get a numerical identity.*

Alternative answer: LHS counts $F_{n,5}$. RHS counts the same thing in a different way. Given a function $f \in F_{n,5}$, some number of elements of $[n]$ map into the subset $[4]$ of the range $[5]$. Let i be that number, and use second counting principle to count $F_{n,5}$ according to what i is. For each i, we need to choose an i-element subset S of $[n]$, make a function from S to $[4]$, and then extend this to a function $f : [n] \to [5]$ by sending every element of $[n] \setminus S$ to 5. Thus the Fundamental Counting Principle says that there are $\binom{n}{i} \cdot 4^i \cdot 1$ functions for each i, so the total number of functions is $\sum_{i=0}^{n} \binom{n}{i} 4^i$.
This alternative answer was longer and not what I had in mind, but a few of you did it this way, and it's kinda fun. Do you see how, expanding on this argument, you could get a different proof of the binomial theorem that applies whenever x and y are nonnegative integers?

Question. Write (and explain) the identity for $S(m, n)$ that comes from considering whether a given set partition has a block $\{m\}$ or not.

Answer: $S(m, n) = S(m-1, n-1) + nS(m-1, n)$
RHS counts partitions of $[m]$ with n blocks using the second counting principle. Partitions with a block $\{m\}$ correspond to partitions of $[m-1]$ with $n-1$ blocks. (The bijection is to take out the block $\{m\}$, and the inverse bijection is to put in the block $\{m\}$.) Partitions without a block $\{m\}$ have the element m in some other block. To count these by the Fundamental Counting Principle, first make a partition of $[m-1]$ with n blocks ($S(m, n)$ ways) and then choose which block to put m into (n ways).
This kind of two-part 2CP argument was a theme that was emphasized in class.

Question. Given nonnegative integers n and k, how many compositions of n have k parts and have the property that every part is ≥ 2?

Answer: The set of compositions of n that have k parts and have the property that every part is ≥ 2 is in bijection with the set of compositions of $n-k$ with k-parts. There are $\binom{n-k-1}{k-1}$ of these. *The bijection is to subtract 1 from each part, and the inverse is to add 1 to each part. This ties in to the "subtract one from each part" idea that was prevalent in a recent homework assignment about partitions.*

Challenge Problems

These ranged from "hard" to "almost impossibly hard." I hope you followed my advice not to spend time on these unless you were completely done working on (and checking) the other problems.

Question. Suppose $k \geq 0$. Find $\lim_{n\to\infty} p_{n-k}(n)$.

Answer: This builds on what you did in Homework 6, Section 1.8., Problem 2 (and related to Problem 13). The Ferrers diagram of a partition of n with $n-k$ parts is a column with $n-k$ boxes and the remaining boxes arranged in a Ferrers shape to the right of that column. When n is much larger than k (as it will be as it goes to ∞), **any** Ferrers shape can be made with those k boxes. So the limit is $p(k)$, the number of partitions of k.

Question. Prove: For an integer $n > 1$, any positive integers $a_1, \ldots, a_n$, and any prime number p, the quantity $\big(\sum_{i=1}^{n} a_i\big)^p - \sum_{j=1}^{n} (a_j)^p$ is a positive integer divisible by p.

Answer: This can be proved by altering the proof of Fermat's Little Theorem in Homework 4, Section 1.7, Problem 11. The point is that you use the Multinomial Theorem to expand $\big(\sum_{i=1}^{n} a_i\big)^p$. By Problem 11a, almost every term in the expansion is divisible by p. There are exactly n terms that are not necessarily divisible by p, and they are $\sum_{j=1}^{n} (a_j)^p$.

Question. Below is a version of Chu's Theorem, written so as to make this problem as easy as possible. Prove it combinatorially by a counting argument which shows that the right side also counts r-subsets of $[n]$.

$$\binom{n}{r} = \sum_{k=1}^{n} \binom{k-1}{r-1}.$$

Answer: For each r-subset of $[n]$, let k be the (numerically) largest element of the r-subset. The right side counts r-subsets according to k. Do you see how? (Think: if I knew the largest element, what would be left to do to make the subset?)

Question. Find the coefficient of q^n in the expression below. (Hint: $\frac{1}{1-x}$ can be expanded as $1 + x + x^2 + x^3 + \cdots$)

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

Comment: This was asking you to make, in the last minutes of the exam, a very large creative and conceptual leap. Possibly the great mathematician who originally did this took more than one class period and had more background information at the time. What can I say? I already said that some of these were almost impossibly hard. This problem will be very important later in the course. I hope you will keep thinking about it in your spare time.