MA/CSC 416
Homework 5, Comments and some solutions.

Section 1.7, Problem 11

Sample solution of Problem 11. The formula for $\binom{p}{r_1,\ldots,r_n}$ (before cancelling) is a fraction whose numerator is a product of integers, one of which equals p. The denominator is a product of integers, all of which are less than p (since all the r_i's are less than p). Since $\binom{p}{r_1,\ldots,r_n}$ is known to be an integer, we know we can cancel all of the integers in the denominator. But p is prime, so the p cannot be canceled from the numerator. This proves part a.

We write the Multinomial Theorem with parameters to match the problem:

$$(x_1 + \cdots + x_n)^p = \sum \binom{p}{r_1,\ldots,r_n} x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n},$$

where the sum is over all n-tuples $(r_1,\ldots,r_n)$ of non-negative integers summing to p. Setting each x_i equal to one gives:

$$n^p = \sum \binom{p}{r_1,\ldots,r_n}.$$

There are exactly n terms on the right side to which part a does not apply: The cases where some r_i equals p and the others are zero. In each of these cases, the multinomial coefficient evaluates to 1. Subtracting off these n terms, we obtain:

$$n^p - n = \sum \binom{p}{r_1,\ldots,r_n},$$

where the sum on the right is now over those tuples $(r_1,\ldots,r_n)$ with each r_i strictly less than p. By part a, each of these terms is divisible by p, so their sum is divisible by p. □

Some of you wrote incorrect proofs for part a as follows:

Incorrect solution of Problem 11a.

$$\binom{p}{r_1,\ldots,r_n} = \frac{p!}{r_1!r_2!\cdots r_n!} = p\frac{(p-1)!}{r_1!r_2!\cdots r_n!}.$$

Therefore $\binom{p}{r_1,\ldots,r_n}$ is a multiple of p. □

If you intend to write a lot of proofs in your life, you should set up a "warning bell" that goes off in your brain in situations like this. In this case, the warning bell would say "Wait! I didn't use the fact that p is prime, and I also didn't use the fact that each r_i is less than p." That means that your "proof" "works" even if p is not prime and even if one of the r_i equals p. That in turn means that your "proof" can be used to "prove" a lot of things that aren't true. So your proof can't be true.

Here are some examples of false things that you can "prove" using exactly the argument some of you used.

Incorrect proof of a non-fact.

$$\binom{6}{3,3} = \frac{6!}{3!3!} = 6\frac{5!}{3!3!}.$$

Therefore $\binom{6}{3,3}$ is a multiple of 6. □

Incorrect proof of a non-fact.

$$\binom{5}{0,5,0} = \frac{5!}{0!5!0!} = 5\frac{4!}{0!5!0!}.$$

Therefore $\binom{5}{0,5,0}$ is a multiple of 5. □

In both cases, the problem is that the fraction left after factoring out the "p" (i.e. 6 or 5 in the incorrect proofs) is not an integer. But when p is prime and each r_i is less than p, you can show that the remaining fraction is an integer, by giving a similar argument to what I said in the correct proof.

Section 1.7, Problem 12

If you're reading the book, you'll know what Merris means by "the two-step inductive proof."

Sample solution 1 of Problem 12. Take as an inductive hypothesis that the binomial theorem has already been proven for the previous case and write $(x+y)^n = (x+y)^{n-1}(x+y)$, which by induction (followed by algebraic manipulations) equals

$$\begin{aligned}\left[\sum_{r=0}^{n-1}\binom{n-1}{r}x^r y^{(n-1)-r}\right](x+y) &= \left[\sum_{r=0}^{n-1}\binom{n-1}{r}x^{r+1}y^{n-(r+1)}\right] + \left[\sum_{r=0}^{n-1}\binom{n-1}{r}x^r y^{n-r}\right] \\ &= \left[\sum_{r=1}^{n}\binom{n-1}{r-1}x^r y^{n-r}\right] + \left[\sum_{r=0}^{n-1}\binom{n-1}{r}x^r y^{n-r}\right].\end{aligned}$$

Separate out the $r = n$ term (which equals x^n) from the first sum and the $r = 0$ term (which equals y^n) from the second sum and combine the sums, then use Pascal's Relation:

$$\begin{aligned}y^n + \left[\sum_{r=1}^{n-1}\left(\binom{n-1}{r-1}+\binom{n-1}{r}\right)x^r y^{n-r}\right] + x^n &= y^n + \left[\sum_{r=1}^{n-1}\binom{n}{r}x^r y^{n-r}\right] + x^n \\ &= \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r} \quad \square\end{aligned}$$

Section 1.7, Problem 19

The underlying fact that was helpful for this proof: If n has the prime factorization $p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, then an integer d is a divisor of n if and only if it can be written as $p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$ with $0 \le s_i \le r_i$ for each $i \in [k]$.

Maybe it would help to rewrite

$$\prod_{t=1}^{k}\left(\sum_{s=0}^{r_t} p_t^s\right) = (1+p_1+p_1^2+\cdots+p_1^{r_1})(1+p_2+p_2^2+\cdots+p_2^{r_2})\cdots(1+p_k+p_k^2+\cdots+p_k^{r_k}).$$

One proof asks: How do we multiply out this product? We don't apply the multinomial theorem, but we reason in exactly the same way we reasoned to prove the multinomial theorem.

Sample solution 1 of Problem 19. The product is the sum of all terms that arise as

$$\text{(one term from the first sum)}\cdot\text{(one term from the second sum)}\cdots\text{(one term from the }k^{\text{th}}\text{ sum)}.$$

But picking one term from each sum like this happens to be the same as choosing, for each p_i, a number of powers (between zero and r_i) of p_i to multiply in. But that's exactly the same as choosing a divisor of n. Thus when we multiply the product out, we get the sum of all divisors of n. $\square$

Another proof is less like the proof we did of the multinomial theorem, but just as good.

Sample solution 2 of Problem 19. The sum of all divisors of n is

$$\sum_{\substack{(s_1,s_2,\ldots,s_k)\\ 0\le s_i\le r_i\ \forall i\in[k]}} p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k} = \sum_{s_1=0}^{r_1}\sum_{s_2=0}^{r_2}\cdots\sum_{s_k=0}^{r_k} p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$$

which can be further rewritten

$$\sum_{s_1=0}^{r_1} p_1^{s_1}\sum_{s_2=0}^{r_2} p_2^{s_2}\cdots\sum_{s_k=0}^{r_k} p_k^{s_k} = \left(\sum_{s_1=0}^{r_1} p_1^{s_1}\right)\left(\sum_{s_2=0}^{r_2} p_2^{s_2}\right)\cdots\left(\sum_{s_k=0}^{r_k} p_k^{s_k}\right) = \prod_{t=1}^{k}\left(\sum_{s=0}^{r_t} p_t^s\right). \quad \square$$

Section 1.8, Problem 2

Each of these parts was pretty straightforward, and hopefully helped you to get a feel for partitions. Here are some sample solutions. Alternatively, it was possible to prove these using the recursion $p_m(n) = p_{m-1}(n-1) + p_n(n-m)$. And in fact, all of them follow from Problem 13, as I explain below.

Part a. To make the Ferrers diagram for a partition of n with $n-2$ parts, first place a column with $n-2$ boxes. There are two boxes remaining, and I can either make them into a second column of size 2 or a second and third column of size 1 each. (Since $n \geq 4$, we know that $n-2 \geq 2$, so even if I place the last two boxes in one column, I still get a valid Ferrers diagram.)

Part b. To make the Ferrers diagram for a partition of n with $n-3$ parts, first place a column with $n-3$ boxes. There are three boxes remaining, and I can either make them into a second column of size 3, or a second and third column of sizes 2 and 1, or three more columns of size 1 each.. (Since $n \geq 6$, we know that $n-3 \geq 3$, so even if I place the last three boxes in one column, I still get a valid Ferrers diagram.)

Part c. The only new information required in this part, beyond Parts a and b is that $p_{n-1}(n) = p_n(n) = 1$ whenever $n \geq 6$. But in fact, $p_n(n) = 1$ whenever $n \geq 1$ and, as long as $n \geq 2$, there is exactly one partition of n with exactly n parts. (It is $[2, 1, \ldots, 1]$.)

Part d. The easiest approach is this:

Sample solution 1 of Problem 2d. To make a partition of n with 2 parts, first choose the second part (which must be at least 1). Since the first part has to be bigger than the second part, the second part is at most $\frac{n}{2}$. Thus the second part is any integer from 1 to $\lfloor \frac{n}{2} \rfloor$. There are $\lfloor \frac{n}{2} \rfloor$ choices. $\square$

You could also argue by induction, although it's not my favorite approach to this problem. I'm including a sample proof by induction so I can discuss an important point.

Sample solution 2 of Problem 2d. We argue by induction. If $n = 1$ or 2, then the identity says $p_2(1) = \lfloor \frac{1}{2} \rfloor$ (i.e. $0 = 0$) or $p_2(2) = \lfloor \frac{2}{2} \rfloor$ (i.e. $1 = 1$). Suppose $n > 2$. Theorem 1.8.7 says that $p_2(n) = p_1(n-1) + p_2(n-2)$. But $p_1(n-1) = 1$, so $p_2(n) = 1 + p_2(n-2)$, which by induction equals $1 + \lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. $\square$

It was absolutely essential that I checked two base cases! Why? Because my induction "reached down" not to $n-1$ but to $n-2$. So if I didn't have two base cases, it would be possible to "prove" something that was only true for every other integer. Here's an example of what can go wrong.

Sample incorrect proof of an incorrect statement. I will prove by induction that every integer is even. As a base case, 0 is even. In general $n = (n-2) + 2$, and since $n-2$ is even by induction and 2 is obviously even, we see that $(n-2) + 2$ is even. ("Even plus even equals even.") So n is even. $\square$

Section 1.8, Problem 6

Think of these compositions as words with a specified number of each letter. The answers are certain multinomial coefficients: $\binom{8}{3,1,4}$, $\binom{8}{5,3}$, and $\binom{8}{1,1,1,1,1,1,1,1} = 8!$.

Section 1.8, Problem 9

There was a pretty specific hint in the back of the book. Let $k = \lfloor \sqrt{n} \rfloor$. The point was, that you needed to make a one-to-one map from subsets of $[k]$ to partitions of n. If you can make such a map then there must be at least as many partitions of n as there are subsets of $[k]$. (Make sure you understand this point.)

The only things left out of the hint were: first, proving that $\pi(S)$ really is a partition—the key point being that the part they define as the first part really was greater than or equal to the other parts—and second, proving that the map was one-to-one.

Section 1.8, Problems 2abc, 13 and 14

Yes, I know I already gave answers to Problem 2, but I'm including it here because it's so close to 13 and 14. Specifically, all parts of Problem 14 and all parts except (d) of Problem 2 follow from the identity in Problem 13. Almost every correct proof that any of you gave for 2a, 2b, or 2c or any part of 14 was just a version of either the inductive proof of 13 or the Ferrers-diagram proof of 13. (Do you believe me? Look at what you did.)

Sample solution of Problem 13a. The case $m = 1$ is the assertion that $p_1(n) = p_1(n+1)$, which is true because both quantities equal 1. Suppose $m > 1$ and make the inductive assumption that the identity has been proved for $m-1$. Then the inductive assumption implies that

$$\sum_{k=1}^{m} p_k(n) = \left(\sum_{k=1}^{m-1} p_k(n)\right) + p_m(n) = p_{m-1}(n+m-1) + p_m(n).$$

On the other hand, by Theorem 1.8.7, $p_m(n+m) = p_{m-1}(n+m-1) + p_m(n)$ as well. □

Sample solution of Problem 13b. I will describe a bijection between Ferrers diagrams of partitions of $n+m$ with exactly m parts and Ferrrers diagrams of partitions of n with at most m parts.

The Ferrers diagram D of a partition of $n+m$ with exactly m parts has exactly m boxes in its first column. The remaining n boxes are located to the right of the first column in a top- and left-justified manner in m or fewer rows. In other words, the remaining boxes form the Ferrers diagram E for a partition of n with at most m parts. The bijection maps D to E.

On the other hand, given a Ferrers diagram E for a partition of n with at most m parts, the fact that E has at most m parts means that we can adjoin a new first column, with m boxes onto E to obtain a new Ferrers diagram D representing a partition of $n+m$ into exactly m parts. The map sending E to D is the inverse of the map described above. □

Problem 2a. Substitute 2 for n and then $n-2$ for m in the identity $\displaystyle\sum_{k=1}^{m} p_k(n) = p_m(n+m)$ to obtain $\displaystyle\sum_{k=1}^{n-2} p_k(2) = p_{n-2}(n)$. But $p_1(2) = p_2(2) = 1$ and $p_k(2) = 0$ when $k > 2$. So when $n-2 \geq 2$ (i.e. $n \geq 4$) this becomes $2 = p_{n-2}(n)$.

Problem 2b. Substitute 3 for n and then $n-3$ for m in the identity $\displaystyle\sum_{k=1}^{m} p_k(n) = p_m(n+m)$ to obtain $\displaystyle\sum_{k=1}^{n-3} p_k(3) = p_{n-3}(n)$. But $p_1(3) = p_2(3) = p_3(3) = 1$ and $p_k(3) = 0$ when $k > 3$. So when $n-3 \geq 3$ (i.e. $n \geq 6$) this becomes $3 = p_{n-3}(n)$.

Problem 2c. Parts a and b accomplish most of this. Now notice (either directly or by a similar argument to that of parts a and b) that $p_{n-1}(n) = p_n(n) = 1$.

Problem 14a. Substitute $n-m$ for n in the identity $\sum_{k=1}^{m} p_k(n) = p_m(n+m)$.

Problem 14b. Substitute n for m in the identity $\sum_{k=1}^{m} p_k(n) = p_m(n+m)$ and notice that all partitions of n have at most n parts, so that the sum on the left equals $p(n)$.

Problem 14c. Substitute $n+m$ for m in the identity $\sum_{k=1}^{m} p_k(n) = p_m(n+m)$. Again, the sum on the left is $p(n)$.

Section 1.8, Problem 18

You should have found $p(9) = 30$ and $p(14) = 135$, both divisible by 5. Hopefully the experience of writing out a lot of partitions helped give you a feel for partitions and for the recursive formula we proved.

Section 1.9, Problem 3

There was nothing to this problem besides understanding what Theorem 1.9.2 says. The "long way around" would be to find the roots—that's the hard part—and then calculate each elementary symmetric function. But the point is, you don't have to actually do any work at all. Just look at the coefficients (and pay attention to extra minus signs). Careful, though... you need a *monic* polynomial to apply the theorem, so you get an extra factor of 2 in one of these cases.