MA/CSC 416

Homework 4, Comments and some solutions.

General comments

Don't fake it. Don't try to bluff your way through a mathematical proof.

Read math. If you want to be able to write proofs, you should be able to read and understand proofs. The only way to do it is to dive in and try, and then get your questions answered when you have questions. Expect to have to read very slowly and think about what each sentence means. If the proofs in the book seem like gibberish to you, then you will, at best, be able to produce gibberish proofs.

Not every proof in the book is at the level of a proof that I would expect you to write. But you have another source of mathematical reading material: Every week I produce these online solutions. These have the added advantage of being about questions that you have already thought about. Are you taking the time to really read them? You should still expect to have to read very slowly and think about what each sentence means. But if you put in that time to really understand the proofs, you will gain an advantage when you try to understand a new math problem and when you try to write your own proofs.

Numbers or sets? Many of you are using numbers and sets interchangeably. So maybe you talk about "a bijection from $\binom{n}{k}$ to $\binom{n}{n-k}$" or you define a_n to be the number of compositions of n and then you say "$a_2 = 11, 2$." When you do that, the only way that I can see that you know what you're talking about is because I already know how to do the problem (and sometimes I still can't understand what you're trying to say). Numbers are numbers and sets are sets. (If you want to be really technical, you could say that actually, numbers are sets, but this isn't a class on constructing number systems using set theory, and if you want to do that, I will make you write all your numbers correctly in terms of set theory, and you'll hate that.)

Learn the terminology. We have learned some terms for combinatorial objects. Not really very many. Subsets, permutations, compositions, partitions, and probably at most 3–4 more. Please learn these terms and use the correct ones. If you are talking about subsets and you say permutations, or if you say partitions when you mean compositions, it's impossible to understand what you are writing.

Section 1.6, Problem 4

a. $4^7 = 16384$.

b. $\binom{4+7-1}{7} = \binom{10}{7} = 120$.

c. $P(4,7) = 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 \cdot (-1) \cdot (-2) = 0$. (This comes out of the formula, or you could just observe that it's impossible to choose 7 elements from a set of 4 elements without replacement.)

d. $\binom{4}{7} = 0$. (Again, it's impossible to choose 7 elements from a set of 4 elements without replacement, even if order doesn't matter. Said another way: A set of 4 elements doesn't have any subsets containing 7 elements.)

Section 1.6, Problem 7

Here the left side (by definition) is the number of words of length n with $n - r$ copies of some letter (I'll choose "$\cdot$") and one copy each of r other letters (I'll choose $1, 2, \ldots, r$). The right side counts ordered selections of r elements, without replacement, from some n-element set (I'll choose $[n]$).

The bijection is readily understood from an example: Take $n = 9$ and $r = 5$ and consider the 9-letter word $\cdot 4 \cdot \cdot 23 \cdot 51$. Think of this word as a way of filling 5 of the 9 "blanks" and leaving 4 blanks unfilled, and recording the order in which you filled blanks. Specifically, interpret the "1" in position 9 as meaning that I chose the 9th blank first. Then I chose the 5^{th} blank (because there is a

"2" in position 5), then the 6^{th} blank, the 2^{nd} blank and the 8^{th} blank. This is the ordered selection 95628. Here is how to go backwards, starting with the ordered selection 95628 and 9 empty blanks _________. Since 9 is 1^{st} in the ordered selection, put "1" in the 9^{th} position (________1) then "2" in the 5^{th} position (____2___1) because 5 is 2^{nd} in the ordered selection, then ____23__1 then _4__23__1 then _4__23_51 and finally fill the remaining blanks with dots ($\cdot 4 \cdot \cdot 23 \cdot 51$).

Here is a formal proof:

Sample Proof of Problem 7. The left side is the number of words of length n with $n - r$ copies of the letter "·" and one copy each of r the letters $1, 2, \ldots, r$. The right side counts ordered selections of r elements, without replacement, from $[n]$. Let $w_1 \cdots w_n$ be a word as described above, i.e. w_i is the i^{th} letter of the word. Define $\eta(w_1 \cdots w_n)$ to be the ordered selection $a_1 \cdots a_r$ where a_i is the unique subscript such that $w_{a_i} = i$ for each $i \in [r]$. The inverse map starts with an ordered selection $a_1 \cdots a_r$ and defines a word $w_1 \cdots w_n$ by setting w_{a_i} equal to i for each $i \in [r]$ and filling the remaining positions with dots. It should now be obvious that these two maps are inverses. □

What do you think about the sentence "It should now be obvious that these two maps are inverses."?

There were several approaches to the problem that involved "counting the same thing two ways." You could think of these as "bijective proofs" where the bijection is the identity map.

Comment: Sometimes, there is a "right" way to set up the problem that makes it easier to see what's going on. In this problem it was what we chose to call our letters. If I had decided that the LHS was going to count the number of words of length n with $n - r$ copies of the letter L_1 and one copy each of the letters L_2 through L_{r+1}, I wouldn't have been wrong, but it would have been a lot harder to see what I was supposed to do. Some of you asked me about this problem and I helped you get started. (For the rest of you: If you're stuck, feel free to talk to me. But it will work better the earlier you start your homework.)

Section 1.6, Problem 14

Here are three short proofs, each using Corollary 1.6.9, which we didn't talk about in class, but which I was counting on your finding when you read Section 1.6. The third of these is my favorite.

Sample Proof 1 of Problem 14. We argue by induction on n. (The base case $n = 0$ is trivial.) Let $k > 0$ and assume that the statement (with n replaced by n') has been proven for all $n' < n$. We separate solutions $(x_1, \ldots, x_m)$ of $x_1 + \cdots x_m \le n$ into two disjoint sets: The set of solutions to $x_1 + \cdots x_m \le n-1$ and the set of solutions to $x_1 + \cdots x_m = n$. By induction, the former set is counted by $\binom{(n-1)+m}{m}$ and by Corollary 1.6.9, the latter set is counted by $\binom{n+m-1}{n} = \binom{n+m-1}{m-1}$. Applying the second counting principle and Pascal's relation, the total number of soutions is $\binom{n+m-1}{m} + \binom{n+m-1}{m-1} = \binom{n+m}{m}$ □

Sample Proof 2 of Problem 14. We separate $(x_1, \ldots, x_m)$ of $x_1 + \cdots x_m \le n$ into disjoint sets according to the value of $x_1 + \cdots x_m$. For each r from 0 to n, there are $\binom{r+m-1}{r}$ solutions, by Corollary 1.6.9. By the symmetry $\binom{a}{b} = \binom{a}{a-b}$ for binomial coefficients, each of these number $\binom{r+m-1}{r}$ is equal to $\binom{r+m-1}{m-1}$. By the second counting principle, the total number of solutions is $\sum_{r=0}^{n} \binom{r+m-1}{m-1}$, which equals $\binom{n+m}{m}$ by Chu's theorem. □

Sample Proof 3 of Problem 14. Inspired by Corollary 1.6.9, restate the problem as follows: Prove that the set A of nonnegative integer solutions $(x_1, \ldots, x_m)$ to $x_1 + \cdots x_m \le n$ has the same number of elements as the set B of nonnegative integer solutions $(x_1, \ldots, x_{m+1})$ to $x_1 + \cdots x_{m+1} = n$. Given $(x_1, \ldots, x_m) \in A$, define $\eta(x_1, \ldots, x_m) = (x_1, \ldots, x_m, n - (x_1 + \cdots x_m)) \in B$. This is easily seen to be a bijection from A to be with inverse $\eta^{-1}(x_1, \ldots, x_{m+1}) = (x_1, \ldots, x_m)$. □

Section 1.6, Problem 17

A lot of people had trouble with this, so I made it "extra credit". I gave +1 or +2 points or none.

Sample Solution to Problem 17. We argue by induction on n. Assume that, for every $n' < n$, there is a composition of n' into distinct Fibonacci numbers. If n is a Fibonacci number then the desired composition is the 1-tuple (n). (This includes the base case $n = 1$.) Otherwise, let F_k be the largest Fibonacci number less than n. This implies in particular that $F_{k+1} > n$, so that by the definition of the Fibonacci numbers, $F_{k-1} + F_k > n$. The latter can be rewritten as $n - F_k < F_{k-1}$ and since the Fibonacci numbers form an increasing sequence, we deduce that $n - F_k < F_k$.

By induction, there is a composition $(a_1, \cdots, a_m)$ of $n - F_k$ whose parts are distinct Fibonacci numbers. Thus $(F_k, a_1, \cdots, a_m)$ is a composition of n into Fibonacci numbers. Since $F_k > n - F_k$, each part a_i is strictly less than F_k, so the parts of $(F_k, a_1, \cdots, a_m)$ are distinct Fibonacci numbers. □

Section 1.6, Problem 27

The answers are in the back of the book, so I decided not to grade this one. **But** please be aware that I may grade simple problems like this where the answer is in the back and if so, you need to explain the answer to get credit. **Also**, please be sure that you understand this problem, even though I didn't grade it.

The key was to "turn the problem on its ear" and see that we're not picking marbles or numbers, we're picking boxes.

Part a. To put the indistinguishable marbles in the boxes, I have to choose ten boxes. The order doesn't matter and I am allowed to repeat boxes. The answer is the number of unordered selections of 10 elements of $\{A, B, C, D\}$ with replacement: $\binom{10+4-1}{10} = 286$.

Part b. To put the numbers $0, 1, \ldots, 9$ in the boxes, I have to choose ten boxes. This time the order **does** matter because the first box I choose will get the number 0, the second box will get the number 1, etc. I am allowed to repeat boxes. The answer is the number of ordered selections of 10 elements of $\{A, B, C, D\}$ with replacement: 4^{10}.

Section 1.6, Problems 19 and 29

I had a proof in mind for these and the book's hints leads you in another direction. My idea for these problems was to simply show that the numbers of objects in question satisfy the Fibonacci recursion, using that recursion as the *definition* of the Fibonacci numbers. Note that if you do this, since the Fibonacci recursion depends on two previous terms, you need two base cases!

Sample proof 1 of Problem 19. As base cases, consider $n = 1$ and $n = 2$. There is exactly one composition of 1 and two compositions of 2, and none of these have a part larger than 2. Thus $\ell_1 = 1 = F_1$ and $\ell_2 = 2 = F_2$. Now let $n > 2$ and assume that for every $k < n$ the numbers ℓ_k and F_k have been shown to coincide.

Separate the set of compositions of n with parts ≤ 2 into two disjoint subsets: the set A of compositions having first part 1 and the set B of compositions having first part 2. The map $(1, a_1, a_2, \ldots, a_k) \mapsto (a_1, a_2 \ldots a_k)$ is a bijection between A and the set of compositions of $n-1$ with parts ≤ 2. Similarly, the map $(2, a_1, a_2, \ldots, a_k) \mapsto (a_1, a_2 \ldots a_k)$ is a bijection from B to the set of compositions of $n - 2$ with parts ≤ 2. Thus, $|A| = \ell_{n-1}$ and $|B| = \ell_{n-2}$. By the second counting principle, $\ell_n = |A| + |B| = \ell_{n-1} + \ell_{n-2}$. By induction, this is equal to $F_{n-1} + F_{n-2} = F_n$. □

To illustrate what I mean about needing two base cases, consider what the proof above says about the case $n = 3$: It's saying "I know that $\ell_3 = \ell_2 + \ell_1$ and I know $\ell_2 = F_2$ and $\ell_1 = F_2$, so $\ell_3 = F_2 + F_1$, which by definition of the Fibonacci numbers equals F_3." What if I didn't check that $\ell_2 = F_2$ as a base case? My inductive "proof" that $\ell_2 = F_2$ would run: "I know that $\ell_2 = \ell_1 + \ell_0$ and I know $\ell_1 = F_1$ and, well, I'm confused about ℓ_0, because I haven't done that case." (We could fix this by including $n = 0$ as a base case. You need two consecutive base cases for the argument to work.) If I don't fix this, my argument for $n = 3$ would be: "I know that $\ell_3 = \ell_2 + \ell_1$ and I know $\ell_1 = F_1$, but I was already confused when I tried to calculate ℓ_2."

Sample proof of Problem 29. To save words in this proof, call a set *sparse* if it contains no two consecutive integers. There are exactly 2 ($= F_2$) subsets of [1], both of which are sparse. There are 3 ($= F_3$) sparse subsets of [2]. (The only non-sparse subset of [2] is [2] itself.) Now let $n > 2$ and assume that for every $k < n$ the number of sparse subsets of $[k]$ is F_{k+1}. Separate the sparse subsets of $[n]$ into two disjoint collections; those which contain n and those which don't. Sparse subsets of $[n]$ not containing n are exactly sparse subsets of $[n-1]$. By induction, these are counted by F_n. A sparse subset of $[n]$ containing n cannot also contain $n-1$. Thus the map $U \mapsto U \setminus \{n\}$ is a bijection from sparse subsets of $[n]$ containing n to sparse subsets of $[n-2]$. By induction, sparse subsets of $[n-2]$ are counted by F_{n-1}. The second counting principle now says that there are $F_n + F_{n-1} = F_{n+1}$ sparse subsets of $[n]$. □

The book's hints suggest that you do these problems using the non-recursive definition of the Fibonacci numbers given in Problem 19 of Section 1.2. It seems to me that the more fundamental definition of the Fibonacci numbers is the recursion, so I think the above proofs are more enlightening. But here is a proof of Problem 19 along the lines of what the book suggests. (Note that the step of proving that these two definitions of the Fibonacci numbers give the same sequence of numbers is missing. I didn't insist that any of you fill this in, but some of you did.)

Sample proof 2 of Problem 19. Count compositions of n with parts of size ≤ 2 according to how many $2's$ appear. If a composition of n is to have r parts of size 2, then it must have $r \leq \lfloor \frac{n}{2} \rfloor$ and there must be $n - 2r$ parts of size 1. Such compositions are words of length $n - r$ with r copies of the letter 2 and $n - 2r$ copies of the letter 1. There are $\binom{n-r}{r}$ such words. Summing over all possible values of r, the number of compositions of n with parts no larger than 2 is:

$$\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r} = F_n.$$

□

A similar proof of Problem 29 is also possible. (You might quote Example 1.6.15.) Another proof of Problem 29, assuming Problem 19, (or vice versa) would be to look closely at the bijection (from class) between subsets and compositions. What happens if you restrict the map to sparse subsets of $[n-1]$? It's very easy to show that you get a bijection from sparse subsets of $[n-1]$ to compositions of n with parts of size 2 or less.

Section 1.7, Problems 6 and 8

The point of these problems is that *polynomial identities contain a lot of information.* When I say a "polynomial identity", I mean an equation of the form "one polynomial equals another polynomial". There are at least three ways to extract additional information from a polynomial identity, and we'll use this terminology in the sample solutions:

Differentiate both sides. This gives a new polynomial identity. (One could argue about the word "new" here, but at the very least, the information in the identity is put into a different form.)

Specialize variables. Consistently on both sides, replace one or more variables with numbers (or other mathematical expressions).

Extract coefficients. If two polynomials are equal, then for any r, the coefficient of x^r in one polynomial equals the coefficient of x^r in the other. When we write an equation that amounts to "coefficient of x^r in one polynomial equals coefficient of x^r in the other polynomial", we are *extracting coefficients* from the polynomial identity.

Sample proof of Problem 6. The binomial theorem, with one of the variables specialized to 1, says

$$(x+1)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Differentiating both sides, we obtain

$$n(x+1)^{n-1} = \sum_{r=0}^{n} r\binom{n}{r} x^{r-1}.$$

Setting $x = 1$, we obtain

$$n \cdot 2^{n-1} = \sum_{r=0}^{n} r\binom{n}{r}.$$

□

You might worry that we should have been more careful with the $r = 0$ term. But it's fine. The derivative of x^0 really is $0x^{-1} = 0$ as long as $x \neq 0$. We're going to set $x = 1$, so it works.

By the way, you can also set x to be other numbers in the binomial theorem or its derivatives. What would happen?

Sample proof of Problem 8. The usual laws of exponentiation tell us that $(x+1)^m(x+1)^n = (x+1)^{m+n}$. The binomial theorem, (with y specialized to 1 as in Problem 6) lets us write this as

$$\sum_{i=0}^{m} \binom{m}{i} x^i \cdot \sum_{j=0}^{n} \binom{n}{j} x^j = \sum_{r=0}^{m+n} \binom{m+n}{r} x^r.$$

To get a term with x^r on the left, we multiply $\binom{m}{i}x^i$ by $\binom{n}{j}x^j$ for all pairs i, j with $i + j = r$. The left side, as written, does not have any terms with $i > m$ or $j > n$, but if we add in such terms, we are only adding zeros. Thus, extracting the coefficient of x^r on both sides, we obtain

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} = \binom{m+n}{r}.$$

□

Section 1.7, Problem 7

The answers are in the back of the book, so again I decided not to grade this one. See comments on Section 1.6 Problem 27. If you understand the proof of the multinomial theorem, this will make sense. If not, please make sure you ask.