MA/CSC 416

Homework 3, Comments and some solutions.

General comments

When you're proving a formula, don't start with the formula, then change it until it is obviously true. For example, if you're trying to prove that $(n-1)^2 - 1 = n(n-2)$, you **would not** write:

$$\begin{aligned} (n-1)^2 - 1 &= n(n-2) \\ (n^2 - 2n + 1) - 1 &= n^2 - 2n \\ n^2 - 2n &= n^2 - 2n. \end{aligned}$$

Why not? If you write that, you are showing that "if $(n-1)^2 - 1 = n(n-2)$) is a true equation, then $n^2 - 2n = n^2 - 2n$ is a true equation." That's really dangerous. What if you started with a false equation and showed that it implies some true equation? Does that make the false equation true? Specifically, if someone started with the equation $0 = 1$ and manipulated it to obtain $1 = 1$, does that make $0 = 1$ true?

Let's try it:

$$\begin{aligned} 0 &= 1 \\ 0 \cdot 0 &= 1 \cdot 0 \\ 0 &= 0 \\ 0 + 1 &= 0 + 1 \\ 1 &= 1 \end{aligned}$$

Are you now convinced that $0 = 1$? I'm not.

When you are proving a formula algebraically, work with the left side separately and/or work with the right side separately, and turm them both into the same thing. For example:

$$\begin{aligned} (n-1)^2 - 1 &= (n^2 - 2n + 1) - 1 \\ &= n^2 - 2n \\ &= n(n-2). \end{aligned}$$

Section 1.2, Problem 12 combinatorially

The following is more than I required you to write, but it is a complete proof with full details. Not convinced? Talk to me about it.

Sample Combinatorial Proof of Problem 12. Pair each n-element subset $S \subseteq [2n]$ with its complement $[2n] \setminus S$. Since $n > 0$, both S and $[2n]$ are not empty, so $([2n] \setminus S) \neq S$. Thus every subset is paired with some **other** set. Also, when we pair S with $[2n] \setminus S$, we also pair $[2n] \setminus S$ with $[2n] \setminus ([2n] \setminus S) = S$, so we really are making **pairs**, not larger sets. Since the collection counted by $\binom{2n}{n}$ can be grouped into some number of pairs, the number $\binom{2n}{n}$ is even. (It is $2k$, where k is the number of pairs.) □

Section 1.2, Problem 12 by Pascal's Triangle

Sample Proof 1 of Problem 12 using Pascal's Triangle. By Pascal's identity, and then the symmetry $\binom{k}{r} = \binom{k}{k-r}$,

$$\binom{2n}{n} = \binom{2n-1}{n} + \binom{2n-1}{n-1} = \binom{2n-1}{n} + \binom{2n-1}{n} = 2\binom{2n-1}{n}.$$

This is two times an integer, so it is an even integer. □

Sample Proof 2 of Problem 12 using Pascal's Triangle. The row of Pascal's triangle containing $\binom{2n}{n}$ has $2n+1$ entries, from $\binom{2n}{0}$ to $\binom{2n}{2n}$. The sum of the entries is 2^{2n}, which is even, and all the other entries besides $\binom{2n}{n}$ come in pairs $\binom{2n}{k} = \binom{2n}{2n-k}$. Thus the sum of all the other entries besides $\binom{2n}{n}$ is even, and we see that the remaining entry $\binom{2n}{n}$ must also be even. □

Section 1.5, Problem 1

If you did a new induction, you were working too hard. Just use what we already know.

Sample Proof of Problem 1a.

$$2+4+6+\cdots+2n = 2(1+2+3+\cdots+n) = 2\frac{(n+1)n}{2} = (n+1)n.$$

The second equality above was proved in class. □

Sample Proof 1 of Problem 1b.

$$1+3+5+\cdots+(2n-1) = (1+2+3+\cdots+2n)-(2+4+6+\cdots+2n) = \frac{(2n+1)2n}{2}-(n+1)n = n^2.$$

The second equality uses Problem 1a and a result proved in class. □

Sample Proof 2 of Problem 1b.

$$1+3+5+\cdots+(2n-1) = (2-1)+(4-1)+(6-1)+\cdots+(2n-1) = (2+4+6+\cdots+2n)-n = (n+1)n-n = n^2.$$

The second equality above is Problem 1a. □

Section 1.5, Problem 6

Almost all of you got "the point" (and at least 2 of "the points") for this problem. But almost all of you (even some who got full credit) need to work on "the fine points" of writing an inductive proof.

First, a note on the right way to think about induction. You should get **out** of the habit of thinking about induction like this: "I know something for n. I will 'push' my knowledge upwards to prove something for $n+1$." That's not wrong, but a better way to think about it this way: "I want to prove something for n. As a tool in my proof, I can 'reach down' and use the same fact about any smaller number." Of course, this only works if there is some lower limit to "reaching down" (i.e. the base case) where the fact has been proven directly. See if you can detect this "reaching down" thinking in the following:

Sample Proof of Problem 6. If $n=1$ then the statement is trivial. Suppose $n>1$, *and assume that the statement* $1^3+\cdots k^3 = \frac{1}{4}k^2(k+1)^2$ *has been proven for all* $k<n$. Then $1^3+\cdots+n^3$ equals $(1^3+\cdots(n-1)^3)+n^3$ which, by induction, equals $\frac{1}{4}(n-1)^2n^2+n^3$. Factoring out $\frac{1}{4}n^2$ from each term, the latter can be written as $\frac{1}{4}n^2\left((n-1)^2+4n\right) = \frac{1}{4}n^2(n^2+2n+1) = \frac{1}{4}n^2(n+1)^2$. □

The clause in italics in this proof could also have been given as "*and assume that the statement* $1^3+\cdots k^3 = \frac{1}{4}k^2(k+1)^2$ *has been proven for* $k=n-1$." Some inductive proofs will require "*all* $k<n$." In settings where this kind of simple induction is considered routine, the clause in italics will usually be deleted entirely. The reader will know what the inductive assumption should be. NOTE: In the setting of your homework, induction is still new enough to you that you should probably still state explicitly the inductive assumption.

Do you see how this proof seems to "reach down" and use the fact about a previous number? The distinction may seem meaningless now, but as you create more complicated inductive proofs (like a recent proof and some proofs coming up soon in the lectures) the "reaching down" thought-process will be valuable. Also, the "pushing-up" idea will get you into trouble in at least two settings that I can think of (which you may need to consider at some point in your mathematical careers): transfinite induction and induction on partially ordered sets.

Here is another key point about the proof. (I mentioned this in the general comments, but I'm saying it again for emphasis.) Several of you wrote proofs where the body of the proof looked like this (or you wrote a similar proof of the "$n+1$ case"):

$$\begin{aligned}
1^3 + \cdots n^3 &= \frac{1}{4}n^2(n+1)^2 \\
\left(1^3 + \cdots (n-1)^3\right) + n^3 &= \frac{1}{4}n^2(n+1)^2 \\
\left(\frac{1}{4}(n-1)^2 n^2\right) + n^3 &= \frac{1}{4}n^2(n+1)^2 \\
(n-1)^2 n^2 + 4n^3 &= n^2(n+1)^2 \\
n^2\left((n-1)^2 + 4n\right) &= n^2(n+1)^2 \\
(n-1)^2 + 4n &= (n+1)^2 \\
n^2 + 2n + 1 &= n^2 + 2n + 1
\end{aligned}$$

This is problematic: Based on what you **actually wrote**, it appears that, assuming the thing that you are trying to prove, you proved the tautology $n^2 + 2n + 1 = n^2 + 2n + 1$. If you want to structure the proof like this, it is **essential** that you use prose (i.e. words in sentences) to explain what you are doing. ("We will show that the desired equation is equivalent to the equation $n^2 + 2n + 1 = n^2 + 2n + 1$.") Otherwise your reader, seeing you assume the conclusion, should be very suspicious of your proof. Even if you explain yourself in prose, this structure of proof is more conceptually complicated than necessary. The sample proof above starts with the expression on the left and by a few simple steps (including induction) shows that it is equal to the expression on the right. Isn't that the very most straightforward way to show that two things are equal? (In more complicated situations, a proof structured around "showing that the desired equation is equivalent to something else" might be appropriate. Here, it seems to just add confusion.)

Section 1.5, Problem 7

Some of you checked this for some specific values of k, but that's not enough (unless you reason precisely using the fact that these are polynomials, but we won't go there for now). The point that seems to be missed is that $\binom{k}{r}$ is a polynomial in the variable k with degree r, and specifically,

$$\binom{k}{r} = \frac{k(k-1)\cdots(k-r+1)}{r!}.$$

Some of you went to the trouble to derive the identity, which wasn't necessary. Confirming an identity by brute force means just doing the algebra to verify that it's true. Just work with the right side, using the formula above until you obtain k^4.

Sample proof of Problem 7.

$$\begin{aligned}
\binom{k}{1} + 14\binom{k}{2} + 36\binom{k}{3} + 24\binom{k}{4} &= k + 14\frac{k(k-1)}{2} + 36\frac{k(k-1)(k-2)}{6} + 24\frac{k(k-1)(k-2)(k-3)}{24} \\
&= k + 7k(k-1) + 6k(k-1)(k-2) + k(k-1)(k-2)(k-3) \\
&= k + 7k^2 - 7k + 6k^3 - 18k^2 + 12k + k^4 - 6k^3 + 11k^2 - 6k \\
&= (k - 7k + 12k - 6k) + (7k^2 - 18k^2 + 11k^2) + (6k^3 - 6k^3) + k^4 \\
&= k^4
\end{aligned}$$

□

The same comment applies here as in Problem 6. Don't start with the equation you're trying to prove and then show that it implies something true.

Section 1.5, Problem 8

Many of the comments on Problem 6 apply to Problem 8 as well. But Problem 8 was more complicated, prompting us to ask: what is the easiest way to check whether two polynomials are equal? The answer (in many cases): Expand both out and check if they have the same coefficients (like we did in Problem 7). Many of you spent pages and pages carefully trying to work with one polynomial and turn it into the other. That was way more work than was necessary.

It is fine to leave out algebra as long as you clearly state what you are leaving out and as long as what you leave out truly is routine (e.g. expanding a product of polynomials). You should always have done the algebra yourself or on a computer algebra system. Otherwise, it is dishonest (and very embarassing if you happen to be wrong!) to just assert that the algebra works out. Factoring is not always routine (just ask Abel or Galois), but notice that if you organize your proof as in the sample, you never have to factor. All you have to do is expand both polynomials and check if they have the same coefficients.

For Problem 8a, the point was to use the fact verified in Problem 7 and then apply Chu's Theorem. (Unfortunately, it was possible to do the problem by just looking at the suggested formulas from the book. Do you understand where this is coming from?) The sample proof will explain the whole story.

Sample proof of Problem 8a. Using the identity verified in Problem 7, we rewrite the left side:

$$1^4 + \cdots + n^4 = \sum_{k=0}^{n} k^4 = \sum_{k=0}^{n} \left[\binom{k}{1} + 14\binom{k}{2} + 36\binom{k}{3} + 24\binom{k}{4} \right].$$

This finite[1] sum can be rearranged as follows:

$$\sum_{k=0}^{n} \binom{k}{1} + 14\sum_{k=0}^{n} \binom{k}{2} + 36\sum_{k=0}^{n} \binom{k}{3} + 24\sum_{k=0}^{n} \binom{k}{4}.$$

Applying Chu's Theorem to each piece, we obtain $\binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5}$ which equals

$$\frac{(n+1)n}{2} + 14\frac{(n+1)n(n-1)}{6} + 36\frac{(n+1)n(n-1)(n-2)}{24} + 24\frac{(n+1)n(n-1)(n-2)(n-3)}{120}.$$

Expanding and gathering like terms, we obtain

$$-\frac{n}{30} + \frac{n^3}{3} + \frac{n^4}{2} + \frac{n^5}{5}.$$

This is exactly the expression that results from expanding the right side. □

Sample Proof of Problem 8b. If $n = 1$ then the statement is trivial. Suppose $n > 1$, *and assume that the statement* $1^4 + \cdots k^4 = \frac{1}{30}k(k+1)(2k+1)(3k^2+3k-1)$ *has been proven for all* $k < n$. Then $1^4 + \cdots + n^4$ equals $(1^4 + \cdots (n-1)^4) + n^4$ which, by induction, equals

$$\frac{1}{30}(n-1)(n-1+1)(2(n-1)+1)(3(n-1)^2+3(n-1)-1)+n^4 = \frac{1}{30}(n-1)n(2n-1)(3n^2-3n-1)+n^4.$$

Expanding, we obtain $\frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n)$, which is the same as what we obtain when we expand $\frac{1}{30}(n(n+1)(2n+1)(3n^2+3n-1)$. □

Section 1.5, Problem 15

Sample Proof 1 of Problem 15. Suppose there is a bowl of m apples and n oranges. (How do you like the m apples?) There are $\binom{m+n}{r}$ ways to choose r pieces of fruit from the bowl. (We are assuming that you can tell the apples apart from each other and that you can tell the oranges apart from each other.) On the other hand we can break up the ways to pick r pieces of fruit according to how many apples we choose. By FCP, there are $\binom{m}{j}\binom{n}{r-j}$ ways to pick j apples and $r-j$ oranges. Now 2CP says $\binom{m+n}{r}$ is equal to

$$\sum_{j=0}^{r} (\text{\# ways to pick } j \text{ apples and } r-j \text{ oranges}) = \sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j}. \qquad \square$$

If the previous proof sounds a little too informal, how about the following:

Sample Proof 2 of Problem 15. Let A and B be disjoint sets with $|A| = m$ and $|B| = n$. There are $\binom{m+n}{r}$ r-element subsets of $A \cup B$. Each r-element subsets of $A \cup B$ contains some number j of elements from A, with $0 \le j \le r$. By the Fundamental Counting Principle, for each j, there are

[1] Recall that for infinite sums, breaking up the summation into smaller summations might change the sum. But this is a finite sum: we can add up a finite list of numbers in any order without changing the sum.

$\binom{m}{j}\binom{n}{r-j}$ ways to choose j elements from A and $r-j$ elements from B. Thus the total number of r-element subsets of $A \cup B$ also equals

$$\sum_{j=0}^{r} (\#\ r\text{-element subsets of } A \cup B \text{ with exactly } j \text{ elements from } A) = \sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j}. \qquad \square$$