MA/CSC 416
Homework 2, Comments and some solutions.

NOTE: If anything here is unclear, please ask me about it!

General comment # 1

You cannot "bluff" your way through a mathematical argument, and it is dishonest to try. If you don't understand what a certain fact is true, don't make up something that sounds plausible. I'm more likely to award partial credit when you say what you know and then acknowledge that you don't understand the rest.

General comment # 2

As I've discussed several times in class, there is a fundamental difficulty when you try to use natural language (i.e. human language, e.g. English, Spanish, Chinese, as opposed to computer language or formal logical languages) to describe a mathematical argument. Would you believe that one of the most important skills for a mathematician is to wring every possible bit of clarity and precision from human language?

It is very important to think carefully about the words and phrases you write. Do they really mean what they need to mean, or did you just write some words that seemed OK? In many of your papers, I underlined phrases that weren't clear or that didn't seem to say what you really meant.

Problem 9b

This argument is very similar to that argument for Pascal's identity from class. (In fact, if you put $k = 2$ in the proof below, you get our proof of Pascal's identity.) Note that the reason that you *add* when you count the words is that you break the set of words up into **disjoint** subsets. It's not correct to describe this as a "step-by-step process."

Sample solution to Problem 9b. The left side counts words with n letters with k distinct letters $a_1, \ldots, a_k$ appearing, with a_i appearing r_i times for each $i \in [n]$. We will count this set of words another way: break the set of words up into k disjoint subsets, where the i^{th} subset consists of those words whose last letter is a_i. By the second counting principle (2CP), we can count each subset and add.

To count the i^{th} subset, notice that making a word (with a_i appearing r_i times) with n letters ending in a_i is the same as making a work with $n-1$ letters (a_i appearing $r_i - 1$ times, each other a_j appearing r_j times) and then tacking on a_i at the end. Thus the i^{th} subset has $\binom{n-1}{r_1,\cdots,r_{i-1},r_i-1,r_{i+1}\cdots r_k}$ elements. Summing over all i we obtain the right side. □

Problem 10a

Sample solution to Problem 10a. By the FCP, $\binom{n}{k}\binom{k}{r}$ counts pairs (S, T) such that $T \subseteq S \subseteq [n]$ with $|S| = k$ and $|T| = r$. The right side counts the same pairs by FCP another way: In Step 1, we choose T to be any r-element subset of $[n]$. In Step 2, we choose $k-r$ elements of the $(n-r)$-element set $[n] \setminus T$ to make up the remaining elements of S (besides those elements in T). FCP says there are $\binom{n}{r}\binom{n-r}{k-r}$ ways to choose these pairs. Since we counted the same thing two different ways, we conclude that $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$. □

Comment: If it's unclear why the first sentence of that proof is true, here it is: Choosing a pair (S, T) is a two-step process. (Step 1: Choose S and Step 2: Choose T.) In Step 1, we choose a k-element subset S of an n-element set $[n]$, and there are $\binom{n}{k}$ ways to do that. In Step 2, we choose an r-element subset T of a k-element set S, and there are $\binom{k}{r}$ ways to do that (no matter which k-element set S we chose in the first step). FCP says that the number of ways to carry out this process is $\binom{n}{k}\binom{k}{r}$. Moving forward, this kind of thing will be so routine that we won't usually explain it. So if you have questions, it would be good to get them answered now.

Problem 10b

Sample solution to Problem 10b. The left side of Problem 10b is the same as in Problem 10a, so we just need to show that the right side of 10b also counts the pairs (S, T) from that problem. We count the pairs yet a third way: In Step 1, we choose a $(k-r)$-elements subset X of $[n]$ and we will eventually have $X = S \setminus T$. In Step 2, we choose an r-element subset T of the $(n-k+r)$-element set $[n] \setminus X$. Now $(T \cup X, T)$ is the desired pair, and we conclude that $\binom{n}{k}\binom{k}{r} = \binom{n}{k-r}\binom{n-k+r}{r}$. □

Problem 10c

Sample solution 1 to Problem 10c. We first show that the right side counts pairs (S, T) such that $T \subseteq S \subseteq [n]$ with $|T| = r$ (and no condition on the size of S). This is FCP, with Step 1 to pick T and Step 2 to pick some additional elements from $[n] \setminus T$ to be in $S \setminus T$. (This is similar to the right side of Problem 10a.)

Next we show that the left side counts the same set of pairs. We use the the 2CP to break the set of pairs up according to the size of S, and we use j to stand for the size of S. If S has j elements, then the number of pairs (just as in the left side of Problems 10a and 10b) is $\binom{n}{j}\binom{j}{r}$. By the 2CP, the total number of pairs is the sum of these numbers over all possible j. Thus $\sum_{j=0}^{n}\binom{n}{j}\binom{j}{r} = \binom{n}{r}2^{n-r}$. □

Comment: The proof given here for 10c might be the "second generation" proof that one would come up with. In the "first generation," you might start with the right side counting pairs (S, T) such that S is an r-element subset of $[n]$ and T is any subset of $[n] \setminus S$. (This would be a different set S and a different set T from what we used in the proof.) Do you see how this is easier? One could do this with a formal bijective proof, but for this problem, it's really easier to just count the same thing two different ways as above. In other situations, the formal bijective proof is easier.

Here is one more proof that might be called "semi-combinatorial" because it uses several facts that we proved combinatorially but also uses a little bit of algebraic manipulation. If pressed, I will admit that this is not a "combinatorial proof," but I did give credit for it.

Sample solution 2 to Problem 10c. Using Problem 10a (which we proved combinatorially), rewrite the left side as $\sum_{j=0}^{n}\binom{n}{r}\binom{n-r}{j-r}$. Now the factor $\binom{n}{r}$ in the summand does not depend on j, so we can factor it out, and the left side becomes $\binom{n}{r}\sum_{j=0}^{n}\binom{n-r}{j-r}$. Each summand is zero unless $j \geq r$, so this is $\binom{n}{r}\sum_{j=r}^{n}\binom{n-r}{j-r}$, and we can reindex this sum (with $k = j - r$) as $\binom{n}{r}\sum_{k=0}^{n}\binom{n-r}{k}$. We proved combinatorially that the sum (without the factor $\binom{n}{r}$ outside) is 2^{n-r}, so we can finally rewrite this as $\binom{n}{r}2^{n-r}$. □

Problem 17

We could construct such a set Z by a two-step process that satisfies the hypotheses of the Fundamental Counting Principle: Step 1 is to choose r elements from X and step 2 is to choose s elements from Y. Thus the answer is $\binom{n}{r}\binom{m}{s}$.

Problem 26

Here are three sample solutions. The first is very similar to what I did in class. The second is an algebraic proof. (However, it uses Pascal's relation, which has a combinatorial proof that we did in class. So perhaps we could think of Sample Solution 2 as "semi-combinatorial.") The third is a direct FCP argument which, if you think about it, is vaguely the same as the first, taking $S = [n]$ and $x = 1$. There is also a nice proof using the Binomial Theorem. (I accidentally did that in class *before* this homework was due!)

Sample solution 1 to Problem 26. Choose some element $x \in S$ and define a map η on subsets of S:

$$\eta(U) = \begin{cases} U \cup \{x\} & \text{if } x \notin U \\ U \setminus \{x\} & \text{if } x \in U \end{cases}$$

(The notation $U \setminus \{x\}$ denotes the set obtained by deleting x from U.)

Notice that for any set $U \subseteq S$, $\eta(\eta(U)) = U$, because either x is inserted then removed, or x is removed then inserted. Notice also that if U is even then $\eta(U)$ is odd and vice-versa. This gives a bijection between even and odd subsets. (Formally, the map is "η restricted to even subsets" and the inverse map is "η restricted to odd subsets".) □

What does "restricted" mean? It means that you only allow certain elements to be input. The point is that "η restricted to even subsets" is a map that takes an even subset and gives an odd subset. Similarly, "η restricted to odd subsets" is a map that takes an odd subset and gives an even subset. These two maps are inverses because $\eta(\eta(U)) = U$ for any U.

Sample solution 2 to Problem 26. If n is odd then consider the sum

$$\binom{n}{n} + \binom{n}{n-2} + \cdots + \binom{n}{1}$$

which counts odd subsets. Applying Pascal's identity to each term, we obtain

$$\left[\binom{n-1}{n} + \binom{n-1}{n-1}\right] + \left[\binom{n-1}{n-2} + \binom{n-1}{n-3}\right] + \cdots + \left[\binom{n-1}{1} + \binom{n-1}{0}\right]$$

Since $\binom{n-1}{n} = 0$, this is just the number of subsets of a set of size $n-1$. Thus there are 2^{n-1} odd subsets of S. Since there are 2^n subsets of S total, there must also be 2^{n-1} even subsets of S.

If n is even then the proof is almost exactly the same. In this case, the *even* subsets are counted by

$$\binom{n}{n} + \binom{n}{n-2} + \cdots + \binom{n}{0}.$$

Since $\binom{n}{0} = \binom{n-1}{0} = 1$, replace the last term by $\binom{n-1}{0}$ and apply Pascals relation to every other term as before to obtain $\displaystyle\sum_{k=0}^{n-1}\binom{n-1}{k}$, which equals 2^{n-1}. Thus there are 2^{n-1} even subsets of S and 2^{n-1} odd subsets of S. □

By the way, we could have argued a differently in the case where n is odd: If n is odd then the even subsets are counted by

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-1}.$$

Applying the relation $\binom{n}{k} = \binom{n}{n-k}$ to each term, this sum can be rewritten

$$\binom{n}{n} + \binom{n}{n-2} + \cdots + \binom{n}{1},$$

which counts the odd subsets.

Sample solution 3 to Problem 26. We know that there are 2^n total subsets of $[n]$. We will use FCP to show that there are 2^{n-1} odd subsets of $[n]$. We can make an odd subset S of $[n]$, we have n decisions to make: For each i from 1 to n, we must decide if $i \in S$. There are two choices for each decision, except that when we get to n, we have only one choice, because only one choice will give an odd subset. Thus there are 2^{n-1} odd subsets.

To show that there are also 2^{n-1} even subsets, we can make exactly the same argument that we made for odd subsets, or we can just subtract $2^n - 2^{n-1}$ to get 2^{n-1}. □

Problem 28

Sample solution 1 to Problem 28. We first count *ordered* factorizations, i.e. count 2×3 and 3×2 as *different* factorizations of 6. Requiring that the factors be relatively prime is equivalent to requiring that no p_i occurs in both factors. Now choosing an ordered factorization is equivalent to choosing a subset I of $[r]$ such that the left factor is $\prod_{i\in I} p_i^{a_i}$. (The right factor is then $\prod_{i\in[r]\setminus I} p_i^{a_i}$.) There are 2^r possible subsets, but the empty set and the full set must be thrown out, since they lead to one of the factors being 1. This leaves $2^r - 2$ possible ordered factorizations. Each unordered factorization corresponds to two ordered factorizations, so there are $2^{r-1} - 1$ unordered factorizations. □

Sample solution 2 to Problem 28. Requiring that the factors be relatively prime is equivalent to requiring that no p_i occurs in both factors, so we just need to decide which factor to put each $p_i^{a_i}$ in. In any factorization, one of the two factors has $p_r^{a_r}$ in it. Since we only count factorizations in one of the possible orders, we may as well take the second factor to have $p_r^{a_r}$ in it. For each other $p_i^{a_i}$, there are two choices (first factor or second factor) of where to put it, giving 2^{r-1} possibilities. Every one of these possibilities is a valid factorization, except when we put all of the $p_i^{a_i}$ in the second factor, because then the first factor is 1. Subtracting off the only disallowed possibility, we see that there are $2^{r-1} - 1$ unordered factorizations. □