MA/CSC 416
Homework 1, Comments and some solutions.

Take note

I have already seen some clear examples of people writing things that they don't understand, that they have gotten from somewhere else. Writing down something that you don't understand that you got from somewhere else is a violation of academic integrity and is easier to detect than you think. (See the section on Homework in the syllabus for a statement of what is acceptable and what is not.) I have taken note of names, and if this continues, I will be forced to contact the Office of Student Conduct. If you have concerns or questions about what is acceptable, please feel free to contact me.

Expectations and grading system

I am grading each problem on a 0–4 point scale. A "4" means that the answer is correct. A "4∼" means that the answer is basically correct, but there is an important point I want you to consider. When I total the scores, a "4∼" counts as a 4. If there is something mathematically wrong or unclear, the score decreases to a "3," then decreases further as problems with the answer increase. If the answer is astoundingly good—it must literally astound me—then I will score it a "5." Sometimes I will score a multipart problem (or a group of problems) not according to the number of parts it has, but by an estimate of "how many problems" it is worth. For example, on this assignment, I scored Problems 5, 6, 7d and 7f out of 6 points (total), reflecting my judgement these all together are worth about "$1\frac{1}{2}$ problems". On the other hand, each part of Problem 8 was scored as 4 points. When I score anything out of something other than 4, I'll indicate that by writing something like "$\frac{5}{6}$".

I am expecting, as a rule (and perhaps in the "limit") that you will master these problems. This will be reflected in scores of "4" on most problems. If you did not perform that well, you may need a little time to get used to the subject and to my expectations, but please make a goal of mastering every problem from now on.

Note that future assignments will be harder, but hopefully you already know that by the time you see this.

General comment on writing

On this assignment (and on the assignments that follow) I made (and will continue to make) comments on your writing: how clearly and concisely you explained yourself. Even students who did the problem very well received such comments. I don't do this to make you feel bad, but to improve the quality of your mathematical writing. Why bother? Because writing is tremendously important in any technical field. I think you are all quite intelligent. But will that help you in the real world if nobody understands what you are talking about? And even if you already write well enough that everyone knows what you are talking about, writing things in the clearest possible way will make people want to read what you write, and will leave them with the impression: "That's someone who really knows what they are talking about!" In academic math or CS, good writing can be the difference between getting fellowships or not, getting papers published or not, getting jobs or not, and getting tenure or not. In a business environment, you will often be the only person who understands the details of a particular technical issue. Being able to explain the details clearly and concisely makes you a tremendous asset to the company.

If all that doesn't convince you to pay attention to the issue, please be aware that as the course progresses, I will increasingly take writing into account in grading your assignments.

To help with your writing, I am including a lot of sample answers to questions. I intend these as examples of good writing. But "fair-is-fair:" if you don't feel that my sample answers are clear and concise, let me know!

Problem 8

I know that many of you have learned a "recipe" for doing inductive proofs.[1] In this class, I would like for you to move away from the recipe towards proofs using prose. Write for someone who understands the principle of induction but didn't learn your "recipe." Another point: Some of you write something about "$n = k + 1$" that doesn't quite make sense with the rest of what you write. Think about what you're saying, don't just spit out the "induction magic words."

Another important issue is *quantification*, which means declaring the status of a variable when you use it. For us, quantification boils down to the difference between something like "for all n" and "for some n" (or equivalently, "there exists n"). What you are trying to prove is that the given statement is true for all n. If, in trying to write an inductive proof, you assume that the given statement is true for all n, then you are assuming what you are trying to prove, which is a problem. If you assume that what you are trying to prove is true for *some* n and work to show that it is true for the next n, you are doing better. Or, if the quantification of your variables is unclear, it might be hard to know whether your argument makes sense.

Below are sample inductive solutions to the three parts of Problem 8, each in a different style.

Sample solution by induction to Problem 8a. As a base case, when $n = 0$ the formula states that $1 = 1$. Suppose the formula has been proven for $n = k$. Then

$$1 + 2 + 2^2 + \cdots + 2^{k+1} = \left(1 + 2 + 2^2 + \cdots + 2^k\right) + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2^{k+2} - 1.$$

Thus the formula holds for $n = k + 1$ as well, and so by induction, it holds for all n. □

Sample solution by induction to Problem 8b. We proceed by induction on n. In the case $n = 0$, the left side of the formula is an empty sum[2] (and thus equal to 0) and the right side is $1 - 1$. In the case $n > 1$, the left side is $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$, which we rewrite as

$$[1 \cdot 1! + 2 \cdot 2! + \cdots + (n-1) \cdot (n-1)!] + n \cdot n!.$$

By induction on n, this can be written as $[n! - 1] + n \cdot n!$ which equals $(n+1) \cdot n! - 1 = (n+1)! - 1$. □

Sample solution by induction to Problem 8c. Notice that the expression $\dfrac{(2n)!}{2^n}$ can be factored as $\dfrac{[2(n-1)]!}{2^{n-1}} \cdot \dfrac{2n \cdot (2n-1)}{2}$. The second factor equals the integer $n(2n-1)$. The first factor is the original expression, with n replaced by $n-1$. Thus by induction, we can assume that the first factor has already been proven to be an integer. Now $\dfrac{(2n)!}{2^n}$ is a product of integers, and therefore an integer. The base case of the induction is the case $n = 0$, where $\dfrac{(2n)!}{2^n} = 1$. □

There are also some very good non-inductive proofs. If you want to be picky, you can say that all of the proofs below use induction (except the next-to-last one), and I can't really argue with you. (For example, anytime you use "$\cdots$" in a mathematical formula where the length of the expression with the "$\cdots$" depends on n, you are implicitly using induction on n. How?)

Sample solution (without induction) to Problem 8a. For any r, the product $(r-1)(1+r+r^2+\cdots+r^n)$ can be expanded to

$$(r + r^2 + r^3 \cdots + r^{n+1}) - (1 + r + r^2 + \cdots + r^n),$$

and after cancellation, we obtain

$$(r-1)(1 + r + r^2 + \cdots + r^n) = r^{n+1} - 1.$$

The formula in 8a is the case $r = 2$ of the formula above. □

Sample solution (without induction) to Problem 8b. The expression $\sum_{i=1}^{n} i \cdot (i!)$ is a telescoping sum. Notice that for any i, the term $i \cdot (i!)$ is $(i + 1 - 1) \cdot (i!) = (i + 1)! - (i!)$. Thus

$$\sum_{i=1}^{n} i \cdot (i!) = \sum_{i=1}^{n} [(i+1)! - (i!)] = (n+1)! - 1.$$ □

[1] Sometimes the recipe can hide the fact that you may be confused about the concept of induction. If that is the case, please come and talk to me or email me. ("Can we please go over the concept behind inductive proofs?") There's no shame in being confused—I am often confused as I learn new mathematics. But don't stay confused. Ask!

[2] If the "empty sum" business makes you nervous, it is reasonable in this problem to take $n = 1$ as a base case.

Sample solution (without induction) to Problem 8c. $\frac{(2n)!}{2^n}$ is the formula (proved in class) for $\binom{2n}{2,2,\ldots,2}$. That multinomial coefficient counts the elements of a finite set (the set of all words of length $2n$ with each letter occurring exactly twice), so it is an integer. □

Another sample solution (without induction) to Problem 8c.

$$\frac{(2n)!}{2^n} = \frac{[(2n)(2n-2)\cdots(4)(2)]}{2^n}\cdot[(2n-1)(2n-3)\cdots(3)(1)] = (n!)\cdot[(2n-1)(2n-3)\cdots(3)(1)]\,.$$

That's also the answer to "What is the integer?" □

Problem 15

Comment 1. Can you really expect to get credit for an unexplained answer when that answer was given in the back of the book? (And ask yourself a similar question about several other problems.) If the answer is given, you are being graded on the explanation.

Comment 2. I let you get away with writing a lot less than what I wrote below, as long as it was clear that you knew where the answer came from.

Comment 3. Just as important as writing the correct things is: not writing ridiculous things. I saw a lot of ridiculous statements like, for example on 15f:

$$\text{“}10! = 2^8\cdot 3^4\cdot 5^2\cdot 7 = 9\cdot 5\cdot 3\cdot 2 = 270.\text{”}$$

So, what should go in between "$10! = 2^8\cdot 3^4\cdot 5^2\cdot 7$" and "$9\cdot 5\cdot 3\cdot 2 = 270$" instead of an equals sign? Good answer: Anything that makes sense and explains the relationship between one and the other. Best answer: Prose. (A good part of what separates human beings from the rest of the animal kingdom is language. Use words.) See the sample below.

Comment 4. At this point in the course, you should feel very comfortable with the Fundamental Counting Principle and when it applies and when it doesn't. If Problems 15, 17 and/or 18 don't seem very straightforward, then you should be worried enough to come and talk to me about it.

Sample solution to Problem 15f. The prime factorization of 10! is $2^8\cdot 3^4\cdot 5^2\cdot 7$. As in Example 1.1.4 from the text, we can choose a divisor by a process of 4 steps: First choose a number of powers of 2 to include. There are 9 choices $0,1,\ldots,8$. Then choose how many powers of 3 (5 choices), how many powers of 5 (3 choices) and how many powers of 7 (2 choices). By the fundamental counting principle the number of divisors is $9\cdot 5\cdot 3\cdot 2 = 270$. □

Problem 16

Almost all of you who got this proof correct did it in two parts: the "if" and the "only if" separately. But the arguments you made were all easily "reversible." **The point:** This was an excellent situation to do the "if and only if" all in one argument. I'll give two different proofs. I think the first is what your book had in mind, but the second is also very good.

First sample solution to Problem 16. The proof consists of three key observations. The first observation is a straightforward generalization of Example 1.1.4 and Problem 15 (and was discussed in class).

Observation 1. *If an integer n has prime factorization $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ then n has*

$$(a_1+1)(a_2+1)\cdots(a_k+1)$$

positive-integer divisors.

The second observation is immediate from the definition of "odd" (i.e. "not divisible by 2").

Observation 2. *The integer $(a_1+1)(a_2+1)\cdots(a_k+1)$ is odd if and only if each (a_i+1) is odd (or equivalently, if and only if each a_i is even).*

Combining the first two observations, we see that n has an odd number of positive-integer divisors if and only if each a_i is even. The proof is completed by the following simple observation.

Observation 3. *An integer n with prime factorization $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ is a perfect square if and only if each a_i is even.* □

Second sample solution to Problem 16. An integer d is a divisor of n if and only if $\frac{n}{d}$ is an integer, and in this case $\frac{n}{d}$ is also a divisor. Break up the set of positive-integer divisors of n into sets of the form $\left\{d, \frac{n}{d}\right\}$. If any two of these sets intersect, they must be the same set. Each set has two elements except in the case where $d = \frac{n}{d}$ (or in other words $d = \sqrt{n}$). Thus there are an even number of positive-integer divisors if and only if $\sqrt{n}$ is not an integer, or equivalently, if and only if n is not a perfect square. □

Problems 17 and 18

See the comments to Problem 15. Possibly 17 was confusing for reasons other than being confused about FCP. To make a function $f : D \to R$, follow a 4-step process: Decide what $f(d_1)$ is, decide what $f(d_2)$ is, decide what $f(d_3)$ is, decide what $f(d_4)$ is.

Problem 21

It may have been a little unclear what the problem was asking. Based on the examples in the "Hints" section at the back of the book, the book seems to have meant that GRIT had to appear in adjacent positions in the word. To count this, just think of "GRIT" as a single "letter" and there are 6! words using the 6 letters $\{F, U, L, B, H, GRIT\}$.

A more interesting interpretation of this problem is the following: Find all the words using the letters $\{F, U, L, B, R, I, G, H, T\}$ where the letters $\{G, R, I, T\}$ occur in the order GRIT *but not necessarily in adjacent positions in the word.* For example, one of the possibilities would be LB$\underline{\text{G}}$H$\underline{\text{R}}$UF$\underline{\text{IT}}$. So, how do you answer the more interesting interpretation of the question? (This is a question I would expect you to be able to answer, after some thought.)